14. 7. 2017
(revolution)

29 Invariance for forms.

H Hilbert space, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

$a : D(a) \times D(a) \longrightarrow \mathbb{K}$ form

$\operatorname{Re} a(u) \geq 0 \quad \forall u \in D(a)$ (accretive)

$V := (D(a), (\cdot \mid \cdot)_V)$

$(u \mid v)_V := \operatorname{Re} a(u, v) + (u \mid v)_H.$

Hypothesis: V complete

(i.e., a is a closed form).

Rem: $a : V \times V \longrightarrow \mathbb{K}$ is continuous.

Rem: 1. $\|u_n\|_V \leq c \qquad u_n \in V$

$\Rightarrow \exists$ ss $u_{n_k} \longrightarrow u$ in V

i.e. $(u_{n_k} \mid v)_V \longrightarrow (u \mid v)_V \quad \forall v \in V$

2. $u_n \to u$ in H, $u_n \in V$

& $\|u_n\|_V \leq c \Rightarrow$ ~~$u_n \longrightarrow u$~~

$u \in V$ & $u_n \xrightarrow[\text{in } V]{} u$

<u>Rem:</u> $X \hookrightarrow Y$ Banach spaces

$x_n \rightharpoonup x$ in X $\Rightarrow$ $x_n \rightharpoonup x$ in Y

<u>Pf.</u> Let $y' \in Y'$ $\Rightarrow$ $y'|_X \in X'$ $\Rightarrow$

$\langle y', x_n \rangle \to \langle y', x \rangle$. $\square$

<u>Pf of 2.</u> 1: $\exists w \in V$ $\exists$ ss $u_{n_k} \rightharpoonup w$

in V $\Rightarrow$ $u_{n_k} \rightharpoonup w$ in H $\Rightarrow$ $w = u$.

2. Suppose $u_n \not\rightharpoonup u$ in V.

$\Rightarrow$ $\exists v \in V$ $(u_n | v)_V \not\to (u|v)_V$

$\Rightarrow$ $\exists \varepsilon > 0$ $\exists$ ss $|(u_{n_k}|v)_V - (u|v)_V| \geq \varepsilon$

But $\exists$ ss $u_{n_{k_\ell}} \rightharpoonup u$ $\lightning$. $\square$

3. Thus $u_n \in V$, $u_n \to u$ in H

$\operatorname{Re} a(u_n) \leq c$ $\Rightarrow$ $u \in V$ & $u_n \rightharpoonup u$ in V.

(29.1) <u>Lemma.</u> $u_n \in V$ $u_n \to u$ in H

$v_n \in V$, $\|v_n\|_V \le C$

$$\mathrm{Re}\, a(u_n, u_n - v_n) \le 0$$

$$\Rightarrow u \in V \quad \& \quad u_n \to u \text{ in } V.$$

<u>Pf.</u> $\mathrm{Re}\, a(u_n) \le \mathrm{Re}\, a(u_n, v_n) \le M \|u_n\|_V \|v_n\|_V$

$$\|u_n\|_V^2 = \|u_n\|_H^2 + \mathrm{Re}\, a(u_n)$$

$$\le \cancel{\|u_n\|} C' + MC \|u_n\|_V = \alpha + 2\beta \|u_n\|_V$$

$$\Rightarrow \sup_{n \in \mathbb{N}} \|u_n\|_V < \infty$$

$$(\|u_n\|_V - \beta)^2 = \|u_n\|_V^2 - 2\beta \|u_n\|_V + \beta^2$$

$$\le \alpha + \beta^2$$

$$\Rightarrow \|u_n\|_V - \beta \le \alpha + \beta^2$$

$\overset{3.}{\Rightarrow}$ claim. $\square$

Hyp: $\overline{D(a)} = H$

$a \sim A$; i.e.

$D(A) = \{ u \in V : \exists f \in H \quad a(u,v) = (f|v)_H \quad \forall v \in V \}$

$Au := f.$

A is m-sectorial.

$\Rightarrow -A$ generates a C_0-sg T on H.

$C \subset H$ closed, convex, P the minimising projection.

(29.2) <u>Proposition.</u> C invariant $\Rightarrow$

$P D(a) \subset D(a)$.

~~Proof. $(I + tA)^{-1}$~~

~~(29.3) Lemma. $J_t = (I + tA)^{-1} u \longrightarrow u$ in V~~

~~as $t \downarrow 0$.~~

~~Proof. $J_t u + t A J_t u = u$~~ $\quad A J_t u = \dfrac{u - J_t u}{t}$

<u>Proof.</u> Let $u \in V$.

$$R_r = (I + rA)^{-1}$$

$$R_r + rAR_r = I \qquad AR_r = \frac{I - R_r}{r}$$

Let $r_n \downarrow 0$

$$R_{r_n} P u =: u_n \longrightarrow Pu \text{ in } H.$$

$$\operatorname{Re} a(u_n, u_n - u) =$$

$$\operatorname{Re} (A R_{r_n} Pu \mid u_n - u) =$$

$$\operatorname{Re} \frac{1}{r_n} (Pu - R_{r_n} Pu \mid R_{r_n} Pu - u) =$$

$$\operatorname{Re} \frac{1}{r_n} (R_{r_n} Pu - Pu \mid u - R_{r_n} Pu) =$$

$$\operatorname{Re} \frac{1}{r_n} \underbrace{(R_{r_n} Pu - Pu \mid u - Pu)}_{\leq 0} + \operatorname{Re} \frac{1}{r_n} (R_{r_n} Pu - Pu \mid Pu - R_{r_n} Pu)$$

$$\leq - \frac{1}{r_n} \| R_{r_n} Pu - Pu \| \leq 0.$$

$$(29.1) \Rightarrow Pu \in V. \ \square$$

(29.3) <u>Lemma.</u> $R_r u \to u$ in V

$\forall u \in V$.

<u>Proof.</u> $r_n \downarrow 0$ $\quad u_n = R_{r_n} u$

$$\operatorname{Re} a(u_n, u_n - u) = \frac{1}{r_n} \operatorname{Re} (u - R_{r_n} u \mid R_{r_n} u - u)$$

$$\leq 0$$

(29.1) $\Rightarrow$ $u_n \to u$ in V $\square$

(29.4) <u>Theorem.</u> Äqu.:

(i) C invariant

(ii) $PV \subset V$ &

$\operatorname{Re} a(Pu, u - Pu) \geq 0 \quad (u \in V)$

(iii) $\exists\, D$ dense in V $\quad P(D) \subset V$

$\operatorname{Re} a(Pu, u - Pu) \geq 0 \quad \forall u \in D$

(iv) $PV \subset V$ and

$\operatorname{Re} a(u, u - Pu) \geq -\omega \|u - Pu\|^2$

$\forall u \in V$ and some $\omega \in \mathbb{R}$.

<u>Proof</u>. (i) $\Rightarrow$ (ii) $\quad PV \subset V$ by (29.2)

Let $u \in V$.

$$\operatorname{Re} a(R_r Pu, u - Pu) = \frac{1}{r}(Pu - R_r Pu \mid u - Pu)$$
$$\geq 0$$

$r \downarrow 0 \quad \Rightarrow$

$$\operatorname{Re} a(Pu \,;\, u - Pu) \geq 0$$

(since $\quad a(\cdot\,, u - Pu) \in V'$).

(iii) $\Rightarrow$ (ii) $\quad$ Let $u \in V \quad \exists u_n \in D$

$u_n \to u$ in V.

$$\operatorname{Re} a(Pu_n, Pu_n - u_n) \leq 0$$

$Pu_n \to Pu$ in H.

(29.1) $\Rightarrow \quad Pu \in V$ & $Pu_n \to Pu$ in V

$\Rightarrow \quad$ ~~$\lim \|Pu_n\|$~~

$$\operatorname{Re} a(Pu) = \lim \{\operatorname{Re} a(Pu - Pu_n, Pu) + \operatorname{Re} a(Pu_n, Pu)\}$$
$$= \lim \operatorname{Re} a(Pu_n, Pu)$$
$$= \lim \{\operatorname{Re} a(Pu_n, u - Pu_n) + \operatorname{Re} a(Pu_n)\}$$

$$\Rightarrow \quad \operatorname{Re} a(Pu) = \lim_{n\to\infty} \operatorname{Re} a(Pu_n, Pu)$$

$$\leq \underline{\lim}_{n\to\infty} \operatorname{Re} a(Pu_n)^{\frac{1}{2}} \operatorname{Re} a(Pu)^{\frac{1}{2}}$$

$$\Rightarrow \quad \operatorname{Re} a(Pu)^{\frac{1}{2}} \leq \underline{\lim}_{n\to\infty} \operatorname{Re} a(Pu_n)^{\frac{1}{2}} .$$

$$\Rightarrow \quad \operatorname{Re} a(Pu, u - Pu) =$$

$$\operatorname{Re} a(Pu, u) - \operatorname{Re} a(Pu) \geq$$

$$\operatorname{Re} a(Pu, u) - \underline{\lim} \operatorname{Re} a(Pu_n) =$$

$$\lim \operatorname{Re} a(Pu_n, u)$$

$$\parallel$$

$$\lim \left[\operatorname{Re} a(Pu_n, u - u_n) + \operatorname{Re} a(Pu_n, u_n) \right]$$

$$- \underline{\lim} \operatorname{Re} a(Pu_n, u_n)$$

Aero

$$\operatorname{Re} a(Pu, u - Pu) \geq$$

$$\lim \operatorname{Re} a(Pu_n, u_n) - \underline{\lim} \operatorname{Re} a(Pu_n) =$$

$$\overline{\lim} \operatorname{Re} a(Pu_n, u_n - Pu_n) \geq 0 .$$

(ii) $\Rightarrow$ ($\cancel{\text{iii}}$ v)

$$\operatorname{Re} a(u, u-Pu) = \operatorname{Re} a(Pu, u-Pu) + \operatorname{Re} a(u-Pu, u-Pu)$$

$$\geq \operatorname{Re} a(Pu, u-Pu) \geq 0.$$

(iv) $\Rightarrow$ (i) $\quad u \in D(A) \Rightarrow$

$$\operatorname{Re} (Au \mid u-Pu) = \operatorname{Re} a(u, u-Pu)$$

$$\geq -\omega \|u-Pu\|^2.$$

$\overset{\S\, 27}{\Longrightarrow}$ invariance. $\square$

14.7.