Mapping theorems for Sobolev spaces of vector-valued functions
(joint work with Wolfgang Arendt)

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Motivation
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**Lemma**

\[ u \in H^1(0, t; L^2(\Omega)) \Rightarrow u(\cdot)^+ \in H^1(0, t; L^2(\Omega)) \]
\[ D_ju(s)^+ = D_ju(s) \cdot 1_{\{u(s)>0\}} \]

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\[ D_j u(s)^+ = D_j u(s) \cdot 1_{\{u(s) > 0\}} \]


Lemma

\[ u \in H^1(\Omega; H) \Rightarrow P_C u(\cdot) \in H^1(\Omega; H) \]

Question

Let $X$ and $Y$ be Banach spaces, $\Omega \subset \mathbb{R}^d$ open and $1 \leq p \leq \infty$. 
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Suppose $F : X \to Y$ is Lipschitz continuous. Under which circumstances does $F$ induce an Operator (Nemytskii- or superposition operator)

$$W^{1,p}(\Omega, X) \to W^{1,p}(\Omega, Y)$$

$$u \mapsto F \circ u$$

?
A counter example

\[ u : (0, 1) \rightarrow C([0, 1]) \]
\[ u(t)(r) = r - t \]
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\[ \frac{d}{dt} u^+(t)(r) = \begin{cases} 
0, & \text{if } r < t \\
-1, & \text{if } r \geq t 
\end{cases} \]
Theorem (Arendt, K., 2017)

(\ast) holds for all Lipschitz continuous $F : X \to Y$ ($|\Omega| < \infty$ or $F(0) = 0$) iff $Y$ has the Radon-Nikodym property.
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iff

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iff

\(Y\) has the Radon-Nikodym property

Note: \(Y\) has the Radon-Nikodym property iff every Lipschitz continuous (or equivalently every absolutely continuous) function \(f : \mathbb{R} \to Y\) is differentiable a.e.
Rating
Rating

+ full characterization
Rating

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+ many spaces in applications have the Radon-Nikodym property (e.g. Hilbert spaces, reflexive spaces)
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+ many spaces in applications have the Radon-Nikodym property (e.g. Hilbert spaces, reflexive spaces)
+ interesting corollaries
Corollary

Suppose $W^{1,p}(\Omega, \mathbb{R}) \hookrightarrow L^{p^*}(\Omega, \mathbb{R})$. 
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Suppose $W^{1,p}(\Omega, \mathbb{R}) \hookrightarrow L^p(\Omega, \mathbb{R})$, then $W^{1,p}(\Omega, X) \hookrightarrow L^{p^*}(\Omega, X)$ for all Banach spaces $X$. 
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Proof.

$u \in W^{1,p}(\Omega, X)$
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Proof.

$u \in W^{1,p}(\Omega, X)$

$\Rightarrow \|u(\cdot)\|_X \in W^{1,p}(\Omega, \mathbb{R})$
**Corollary**

Suppose \( W^{1,p}(\Omega, \mathbb{R}) \hookrightarrow L^{p^*}(\Omega, \mathbb{R}) \), then \( W^{1,p}(\Omega, X) \hookrightarrow L^{p^*}(\Omega, X) \) for all Banach spaces \( X \).

**Proof.**

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\begin{align*}
  u \in W^{1,p}(\Omega, X) \\
  \Rightarrow \|u(\cdot)\|_X \in W^{1,p}(\Omega, \mathbb{R}) \\
  \Rightarrow \|u(\cdot)\|_X \in L^{p^*}(\Omega, \mathbb{R})
\end{align*}
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Corollary

Suppose \( W^{1,p}(\Omega, \mathbb{R}) \hookrightarrow L^{p^*}(\Omega, \mathbb{R}) \), then \( W^{1,p}(\Omega, X) \hookrightarrow L^{p^*}(\Omega, X) \) for all Banach spaces \( X \).

Proof.

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\begin{align*}
  u & \in W^{1,p}(\Omega, X) \\
  \Rightarrow \| u(\cdot) \|_X & \in W^{1,p}(\Omega, \mathbb{R}) \\
  \Rightarrow \| u(\cdot) \|_X & \in L^{p^*}(\Omega, \mathbb{R}) \\
  \Rightarrow u & \in L^{p^*}(\Omega, X)
\end{align*}
\]
Rating
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- (*)& almost never needed for all $F$
Rating

- (✱) almost never needed for all $F \Rightarrow$ too broad for applications
Theorem (Arendt, K., 2017)

Let $F : X \to Y$ be Lipschitz continuous AND one-sided Gateaux differentiable ($|\Omega| < \infty$ or $F(0) = 0$), then (*) holds.
**Theorem (Arendt, K., 2017)**

Let $F : X \to Y$ be Lipschitz continuous AND one-sided Gateaux differentiable ($|\Omega| < \infty$ or $F(0) = 0$), then $(\ast)$ holds.

Further $D_j(F \circ u(\xi)) = D^+_{D_ju(\xi)} F(u(\xi)) = D^-_{D_ju(\xi)} F(u(\xi))$ a.e.
Examples

Seen before: $\| u(\cdot) \|_X \in W^{1,p}(\Omega, \mathbb{R})$ for all $u \in W^{1,p}(\Omega, X)$
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\[ D_{D_j u(\xi)}^+ \| u(\xi) \|_X = \sup_{x' \in J(u(\xi))} \langle D_j u(\xi), x' \rangle \]

where \( J(x) := \{ x' \in X', \| x' \| = 1, \langle x, x' \rangle = \| x \|_X \} \)
Examples

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D^-_{D_j u(\xi)} \| u(\xi) \|_X = \inf_{x' \in J(u(\xi))} \langle D_j u(\xi), x' \rangle
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Examples

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D^+_{D_j u(\xi)} \| u(\xi) \|_X = \sup_{x' \in J(u(\xi))} \langle D_j u(\xi), x' \rangle
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\]

\[\Rightarrow D_j \| u(\xi) \|_X = \langle D_j u(\xi), x' \rangle\]

for all \( x' \in J(u(\xi)) \)
Examples

Let $X$ be a Banach lattice with order continuous norm and $F : X \to X$ Lipschitz continuous and convex.
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e.g. $X = L^1$ and $F(x) = x^+$

$$D_ju(\cdot)^+ = P_{u^+}(\cdot) D_ju(\cdot)$$
Application

Let $H$ be a separable Hilbert space, $1 < p < \infty$
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$\tilde{B} : D(\tilde{B}) \subset L^p(\Omega, H) \to L^p(\Omega, H)$

$u \mapsto B \circ u$
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$A : D(A) \subset L^p(\Omega, \mathbb{R}) \to L^p(\Omega, \mathbb{R})$ sectorial
Application

Let $H$ be a separable Hilbert space, $1 < p < \infty$

$$B : D(B) \subset H \rightarrow H \text{ sectorial}$$

$$\tilde{B} : D(\tilde{B}) \subset L^p(\Omega, H) \rightarrow L^p(\Omega, H)$$

$$u \mapsto B \circ u$$

$$A : D(A) \subset L^p(\Omega, \mathbb{R}) \rightarrow L^p(\Omega, \mathbb{R}) \text{ sectorial}$$

$$\exists! \tilde{A} : D(\tilde{A}) \subset L^p(\Omega, H) \rightarrow L^p(\Omega, H)$$

$$R(\lambda, \tilde{A})(f \otimes x) = R(\lambda, A)f \otimes x \ \forall \lambda < 0$$
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Theorem (Arendt, K., 2017)

Suppose that $A$ and $B$ have bounded imaginary powers,
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$$\varphi_{bip} A + \varphi_{bip} B < \pi$$
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Suppose that $A$ and $B$ have bounded imaginary powers,

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Further $B$ has compact resolvent
Application

Theorem (Arendt, K., 2017)

Suppose that $A$ and $B$ have bounded imaginary powers,

$$\varphi_{bip}A + \varphi_{bip}B < \pi$$

Further $B$ has compact resolvent and $D(A) \subset W^{1,p}(\Omega, \mathbb{R})$. 
Application

Theorem (Arendt, K., 2017)

Suppose that $A$ and $B$ have bounded imaginary powers,

$$\varphi_{\text{bip}} A + \varphi_{\text{bip}} B < \pi$$

Further $B$ has compact resolvent and $D(A) \subset W^{1,p}(\Omega, \mathbb{R})$.

Then

$$\tilde{A} + \tilde{B} : D(\tilde{A}) \cap D(\tilde{B}) \to L^p(\Omega, H)$$

is sectorial and has compact resolvent.
Reference

arXiv:1611.06161