Exercise 1: Prove the De Morgan's Law

$$(A \cup B)^c = A^c \cap B^c$$

Solution:

$$x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B \Leftrightarrow x \notin A \text{ and } x \notin B \Leftrightarrow x \in A^c \text{ and } x \in B^c \Leftrightarrow x \in A^c \cap B^c$$

Exercise 2: Let $(\Omega, \mathcal{F}, P)$ be a probability space. Show that for all sets $A_1, A_2, \ldots \in \mathcal{F}$ we have that

$$P\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} P[A_i]$$

Solution: Define $B_1 = A_1$ and for $i = 2, \ldots, n$ let $B_i = A_i \setminus \left(\bigcup_{k=1}^{i-1} A_k\right)$. It follows that the B_i are pairwise disjoint and

$$\bigcup_{k=i}^{n} A_i = \bigcup_{k=i}^{n} B_i$$

and therefore

$$P\left[\bigcup_{k=i}^{n} A_i\right] = P\left[\bigcup_{k=i}^{n} B_i\right] = \sum_{i=1}^{n} P[B_i] \leq \sum_{i=1}^{n} P[A_i].$$

The last inequality holds since $P[B_i] \leq P[A_i]$ for all i.

Exercise 3: Let A, B and C be non-disjoint sets of the set Ω. Using only the operators for union, section, difference and complement as well as the letters A, B and C write down expressions for events A, B and C where

a) at least one event is true

b) only the event A is true

c) A and B are true but C is not

d) all events are true

e) none of the events is true

f) exactly one event is true

g) at the most two events are true

Solution: (There may exist more than one solution)

a) $A \cup B \cup C$

b) $A \cap B^c \cap C^c$

c) $A \cap B \cap C^c$

d) $A \cap B \cap C$

e) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

f) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
or
$(A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B))$

g) $(A \cap B \cap C)^c$

Exercise 4: Assume a town where there are only two newspapers Z_1 and Z_2. 60% of the people read Z_1 and 80% read Z_2. 10% neither read Z_1 nor Z_2. Calculate the probability for a randomly chosen person to read

a) both newspapers

b) Z_1 but not Z_2

c) Z_2 under the assumption of not reading Z_1

d) one newspaper at the most

e) not Z_1

Solution: $P[Z_1] = 0.6,\ P[Z_2] = 0.8,\ P[Z_1^c \cap Z_2^c] = P[(Z_1 \cup Z_2)^c] = 0.1 \Rightarrow P[Z_1 \cup Z_2] = 0.9.$

a) $P[Z_1 \cap Z_2] = P[Z_1] + P[Z_2] - P[Z_1 \cup Z_2] = 0.5$

b) $P[Z_1 \cap Z_2^c] = P[Z_1] - P[Z_1 \cap Z_2] = 0.1$

c)
$$P[Z_2|Z_1^c] = \frac{P[Z_1^c \cap Z_2]}{P[Z_1^c]} = \frac{P[Z_2] - P[Z_2 \cap Z_1]}{P[Z_1^c]} = 0.75$$

d) $P[(Z_1 \cap Z_2)^c] = 1 - P[Z_1 \cap Z_2] = 0.5$

e) $P[Z_1^c] = 0.4$

Exercise 5: A human resources manager has for open positions and eight equally qualified applicants, consisting of three women and five men. Calculate the probability of all four positions to be staffed with men. Assume that all candidates have the same probability and are chosen randomly.

Solution: Number of possibilities for choosing four applicants out of eight. Drawing without putting aside (one position for every candidate!) and without considering the order.

$$|\Omega| = \binom{8}{4} = \frac{8!}{(4!)(4!)} = 70$$
$$A = \{\text{All chosen candidates are men}\}$$

Number of possibilities for choosing four men out of five:

$$|A| = \binom{5}{4} = \frac{5!}{(4!)(1!)} = 5$$

$\Rightarrow P[A] = 5/70 = 1/14.$

Exercise 6: Prove the Poincare-Sylvester equation

$$P\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \le k_1 < \ldots < k_i \le n} P[A_{k_1} \cap \ldots \cap A_{k_i}]$$

(Hint: Induction)

Solution: By induction we have:

$n = 2$:

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$$

$n \to n+1$:

$$\begin{aligned}
P\left[\bigcup_{i=1}^{n+1} A_i\right] &= \left[\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right] \\
&= P\left[\bigcup_{i=1}^{n} A_i\right] + P\left[A_{n+1}\right] - P\left[\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right] \\
&= \sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \le k_1 < \ldots < k_i \le n} P\left[A_{k_1} \cap \ldots \cap A_{k_i}\right] + P[A_{n+1}] - \\
&\sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \le k_1 < \ldots < k_i \le n} P\left[A_{k_1} \cap \ldots \cap A_{k_i} \cap A_{n+1}\right] \\
&= \sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \le k_1 < \ldots < k_i \le n} P\left[A_{k_1} \cap \ldots \cap A_{k_i}\right] + P[A_{n+1}] \\
&\sum_{i=2}^{n+1} (-1)^{i+1} \sum_{1 \le k_1 < \ldots < k_{i-1} \le k_i = n+1} P\left[A_{k_1} \cap \ldots \cap A_{k_{i-1}} \cap A_{k_i}\right] \\
&= \sum_{i=1}^{n+1} (-1)^{i+1} \sum_{1 \le k_1 < \ldots < k_i \le n+1} P\left[A_{k_1} \cap \ldots \cap A_{k_i}\right]
\end{aligned}$$