Energy Derivatives

Lecture Notes
Ulm University

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Chapter 1

Fundamentals

1.1 Markets and Price Processes

Since the deregulation of electricity markets in the end of the 1990s, power can be traded at exchanges like the Nordpool or the European Energy Exchange (EEX). All exchanges have established spot and futures markets.

The spot market usually is organised as an auction, which manages the distribution of power in the near future, i. e. one day ahead. Empirical studies, such as Knittel and Roberts (2001) using hourly prices in the California power market, show that spot prices exhibit seasonalities on different time scales, a strong mean-reversion and are very volatile and spiky in nature. Because of inherent properties of electricity as an almost non-storable commodity such a price behaviour has to be expected, see Geman (2005).

Due to the volatile behaviour of the spot market and to ensure that power plants can be deployed optimally, **power forwards and futures** are traded. Power exchanges established the trade of forwards and futures early on and by now large volumes are traded. A power forward contract is characterized by a fixed delivery price per MWh, a delivery period and the total amount of energy to deliver. Especially the length of the delivery period and the exact time of delivery determine the value and statistical characteristics of the contract vitally. One can observe, that contracts with a long delivery period show less volatile prices than those with short delivery. These facts give rise to a a term structure of volatility in most power forward markets, which has to be modelled accurately in order to be able to price options on futures. Figure 1.1 gives an example of such a term structure for futures traded at the EEX. Additionally, seasonalities can be observed in the forward curve within a year. Monthly contracts during winter months show higher prices than comparable contracts during the summer (cp. Figure 1.2).

Aside from spot and forward markets, valuing options is an issue for market participants. While some research has been done on the valuation of options on spot power, hardly any results can be found on options on forwards and futures. Both types impose different problems for the valuation.

Spot options fail most of the arbitrage and replication arguments, since power is almost non-storable. Some authors take the position to find a realistic model to describe the prices of spot prices and then value options via risk-neutral expectations (cp. de Jong and Huisman (2002), Benth, Dahl, and Karlsen (2004), Burger, Klar, Müller, and Schindlmayr (2004)). Other ideas explicitly take care of the special situation in the electricity production and use power plants to replicate certain contingent claims (cp. Geman and Eydeland (1999)).

Forward and futures options are heavily influenced by the length of their delivery period and their time to maturity. In Clewlow and Strickland (1999), for example, a one-factor model is presented, that tries to fit the term structure of volatility, but that does not incorporate a delivery period, since it is constructed for oil and gas markets.

As an example let us have a look at the EEX spot market. Here we have the following structure

- the EEX spot market is a day-ahead auction for single hours of the following day

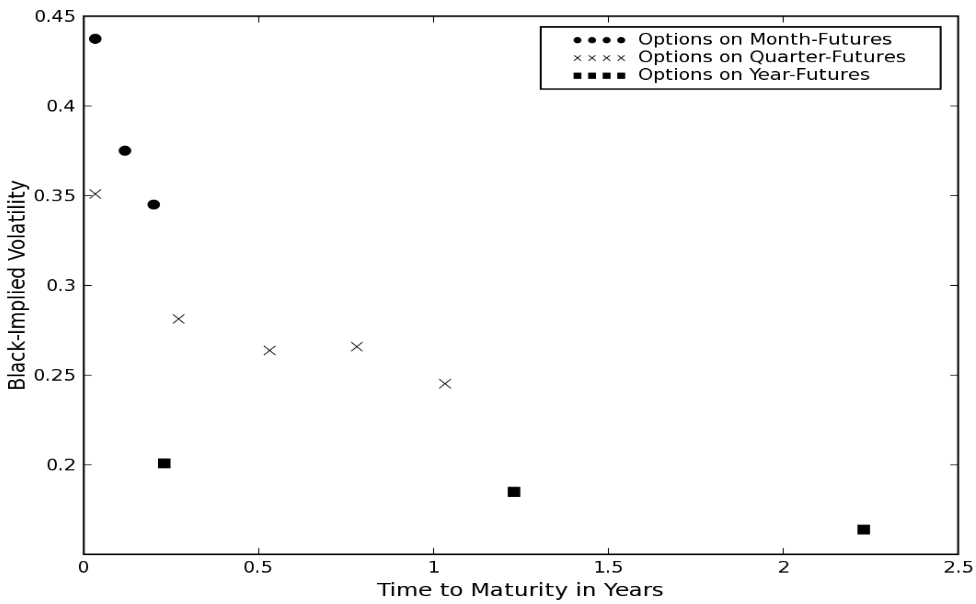


Figure 1.1: Implied volatilities of futures with different maturities and delivery periods, Sep. 14

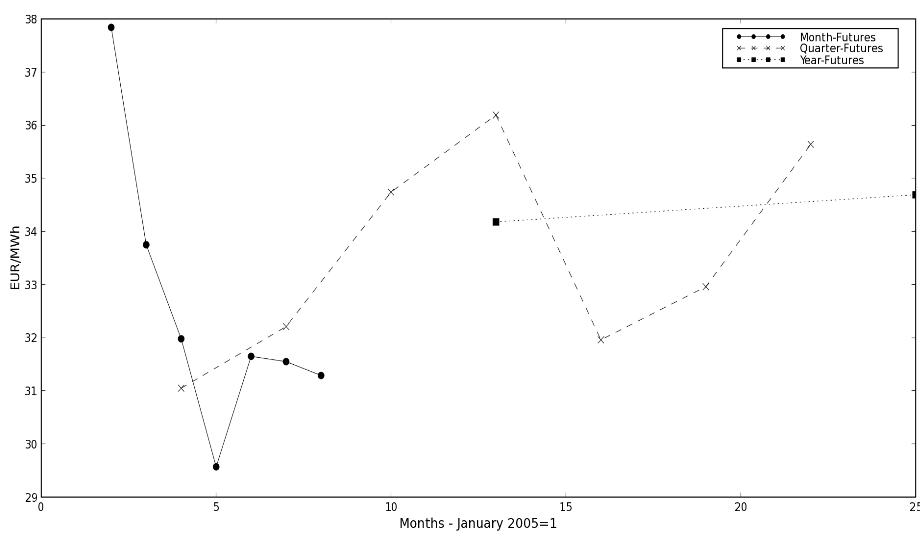


Figure 1.2: Forward prices of futures with different maturities and delivery periods, Feb. 18

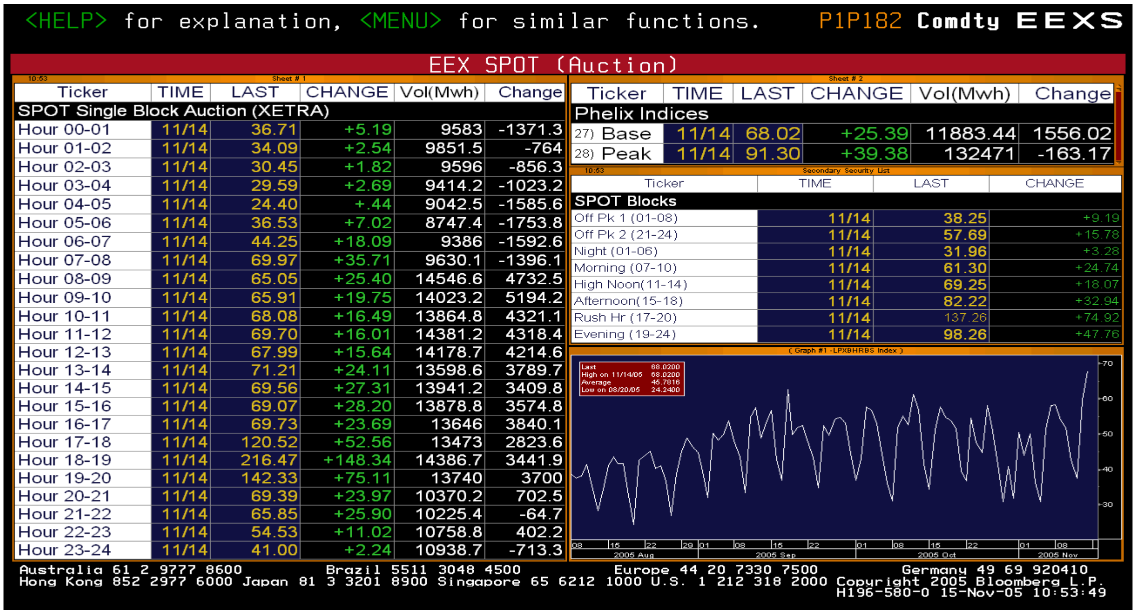
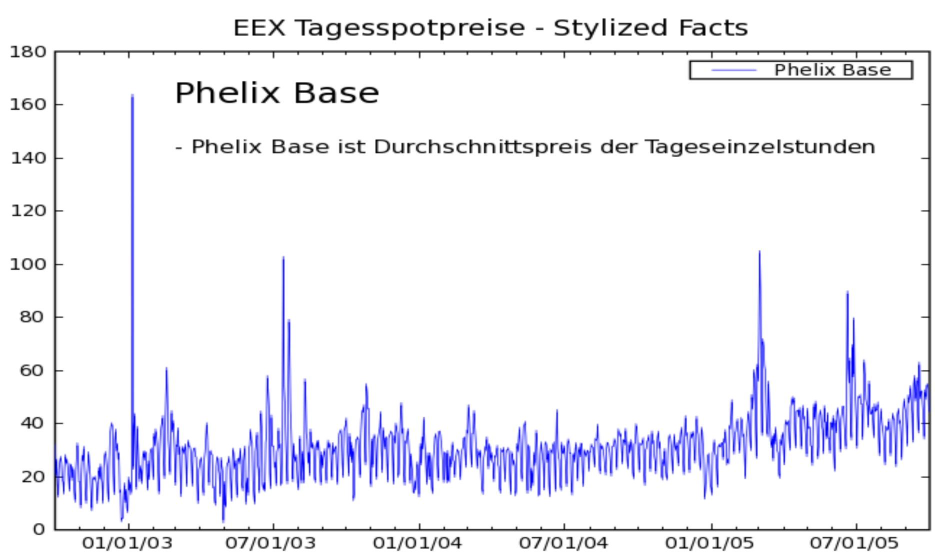
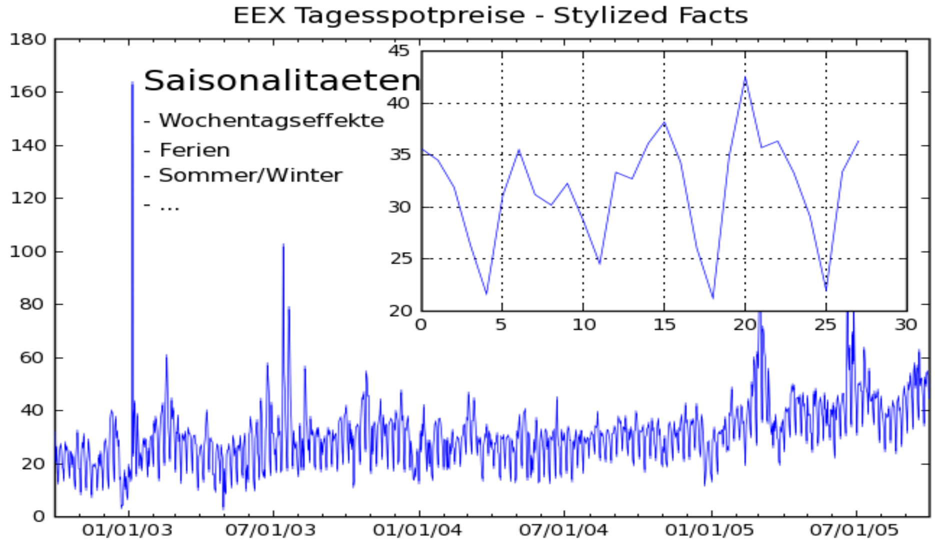


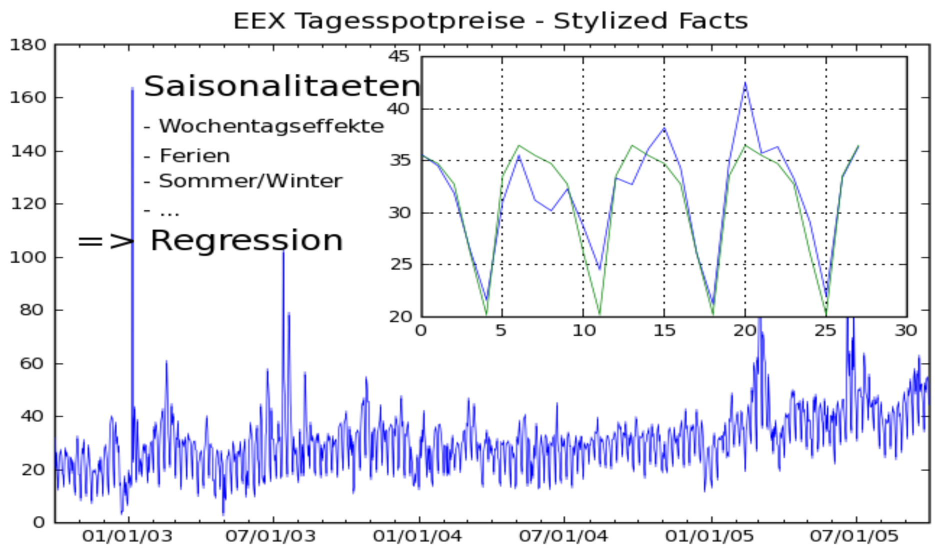
Figure 1.3: Bloomberg screen for energy spot prices

- participants submit their price offer/bit curves, the EEX system prices are equilibrium prices that clear the market.
- EEX day prices are the average of the 24-single hours.
- on fridays the hours for the whole weekend are auctioned.
- similar structures can be found on other power exchanges(Nord Pool, APX, etc.).

The following are examples of price processe



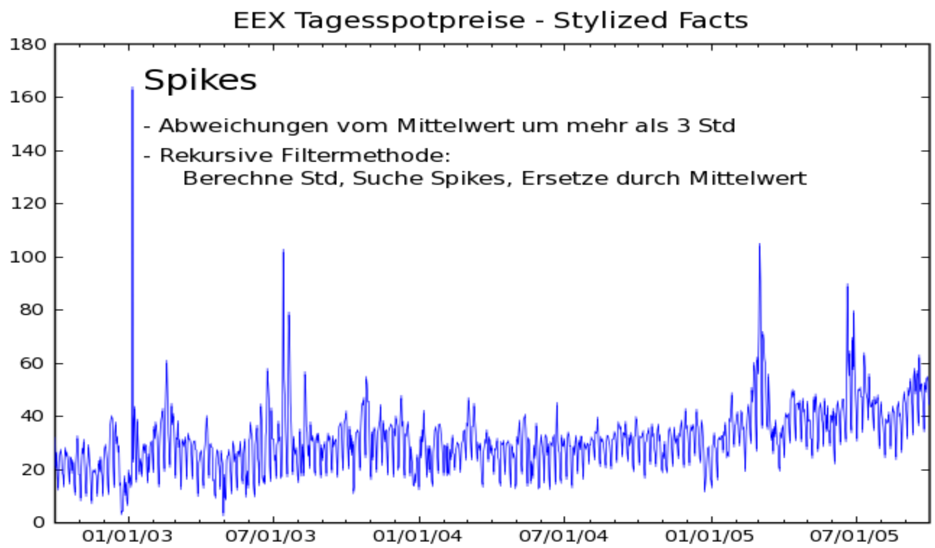


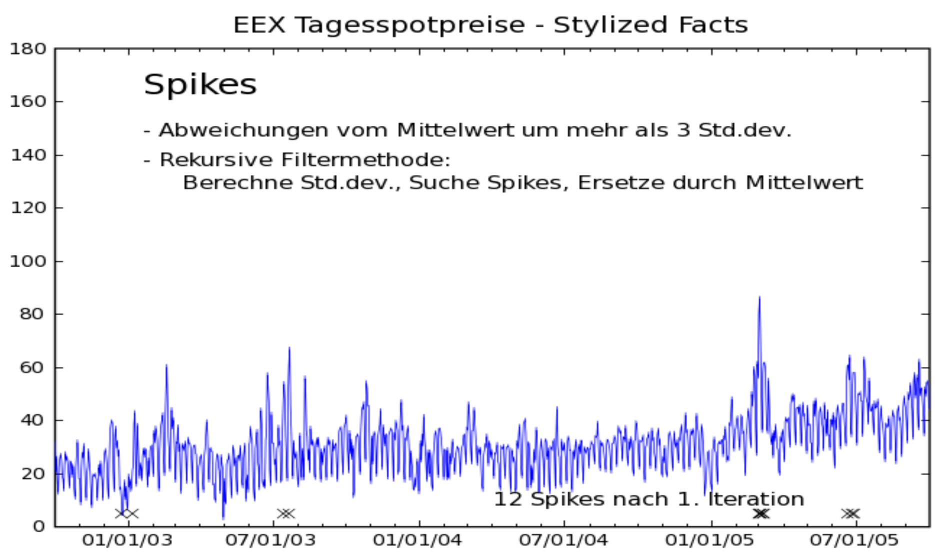


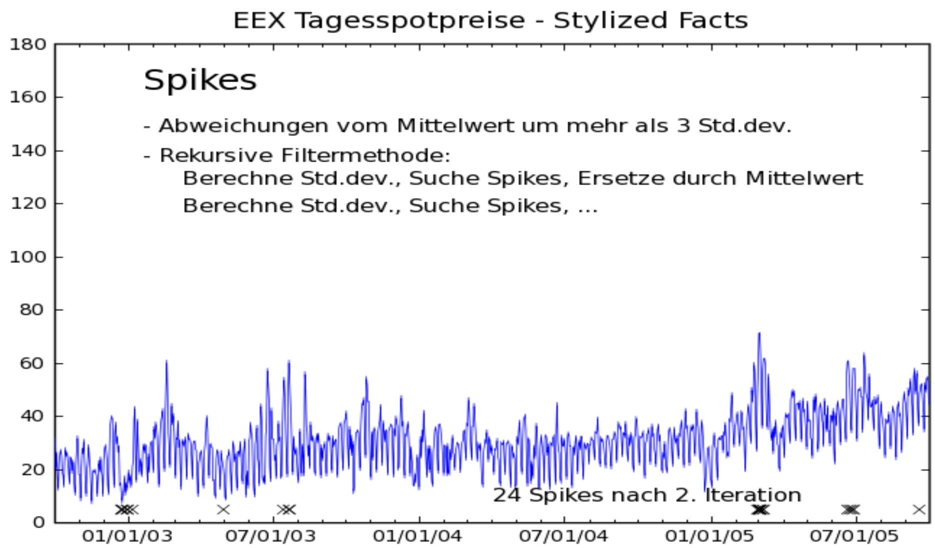
To analyse seasonalities one can preform a regression analysis. This can be done by standard methods assuming a model for the mean, e. g.

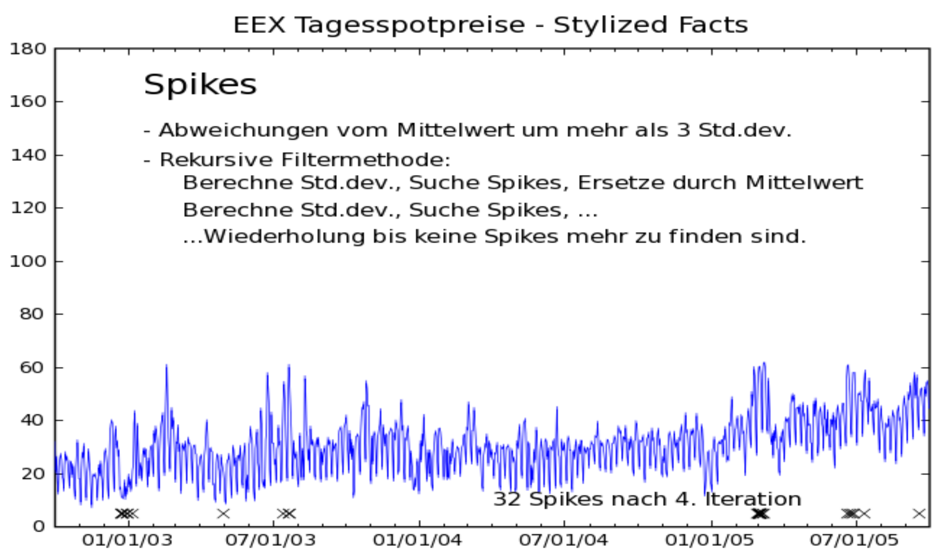
$$\begin{aligned} S_t &= \beta_1 \cdot 1(\text{if } t \in \text{Mondays}) + \ldots + \beta_7 \cdot 1(\text{if } t \in \text{Sundays}) \\ &\quad + \text{ other calendar day effects} \\ &\quad + \beta_8 \cdot t \text{ for long term linear trend} \\ &\quad + \beta_9 \sin\left(\frac{2\pi}{365}(t-c)\right) \text{ for summer/winter seasonality} \\ &\quad + \ldots \end{aligned}$$

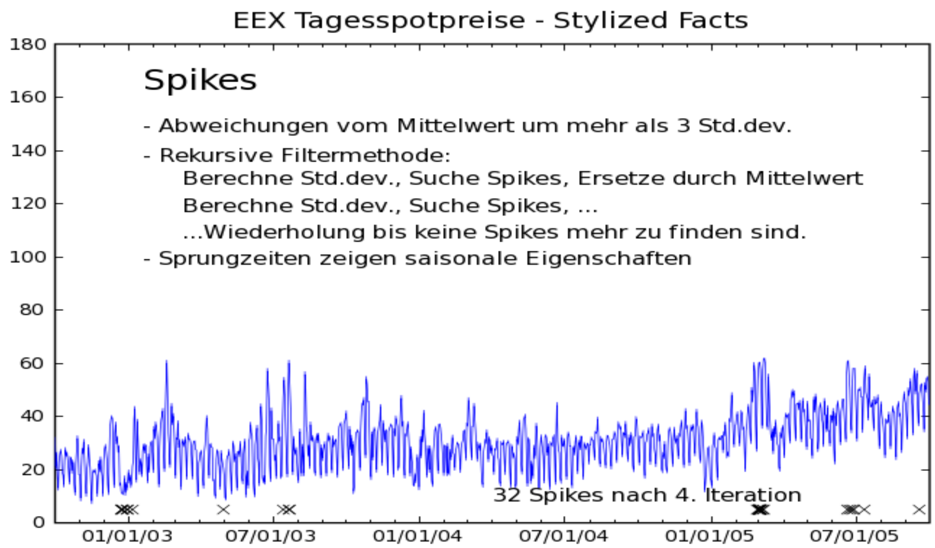
The unknown parameters $\beta_1, \ldots, \beta_p$ can be estimated easily by Least-Squares-Methods.
We also observe spikes


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std.dev.
- Rekursive Filtermethode:
Berechne Std.dev., Suche Spikes, Ersetze durch Mittelwert
12 Spikes nach 1. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std.dev.
- Rekursive Filtermethode:
Berechne Std.dev., Suche Spikes, Ersetze durch Mittelwert
Berechne Std.dev., Suche Spikes, ...
24 Spikes nach 2. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05

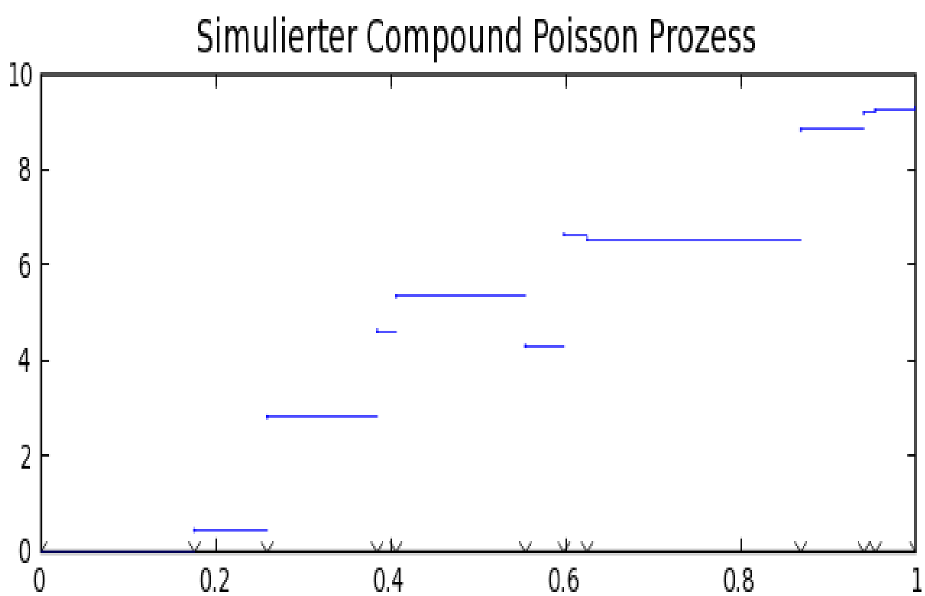

EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std.dev.
- Rekursive Filtermethode:
Berechne Std.dev., Suche Spikes, Ersetze durch Mittelwert
Berechne Std.dev., Suche Spikes, ...
...Wiederholung bis keine Spikes mehr zu finden sind.
32 Spikes nach 4. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05

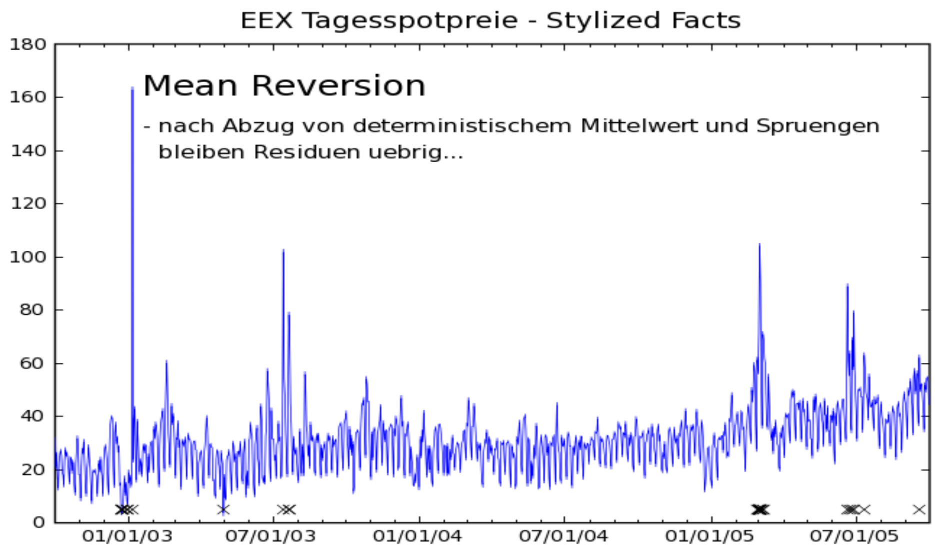


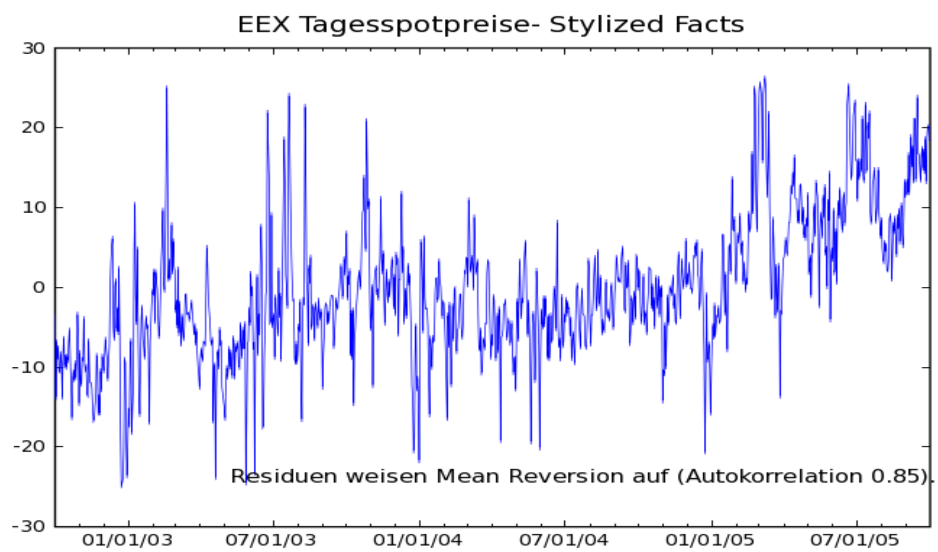
Spikes are often modelled by jumps. The process of jumps is often modelled by a compound poisson process

$$CP_t := \sum_{i=1}^{N_t} J_i$$

N_t is a Poisson process with intensity λ, which randomly jumps by 1 unit, so it counts how many jumps occurred up to time t. J_i is the random jump size of the ith jump.







1.2 Basic Products and Structures

We mostly have been dealing so far with derivatives based on underlying assets – stock – existing, and available for trading, now. It frequently happens, however, that the underlying assets relevant in a particular market will instead be available at some time in the future, and need not even exist now. Obvious examples include crop commodities – wheat, sugar, coffee etc. – which might not yet be planted, or be still growing, and so whose eventual price remains uncertain – for instance, because of the uncertainty of future weather. The principal factors determining yield of crops such as cereals, for instance, are rainfall and hours of sunshine during the growing season. Oil, gas, coals are another example of a commodity widely traded in the future, and here the uncertainty is more a result of political factors, shipping costs etc. Our focus here will be on electricity later on. Financial assets, such as currencies, bonds and stock indexes, may also be traded in the future,

on exchanges such as the London International Financial Futures and Options Exchange (LIFFE) and the Tokyo International Financial Futures Exchange (TIFFE), and we shall restrict attention to financial futures for simplicity.
We thus have the existence of two parallel markets in some asset – the *spot market*, for assets traded in the present, and the *futures market*, for assets to be realized in the future. We may also consider the combined spot-futures market.
We now want briefly look at the most important products.

1.2.1 Forwards

A *forward contract* is an agreement to buy or sell an asset S at a certain future date T for a certain price K. The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position. The settlement date is called *delivery date* and the specified price is referred to as *delivery price*. The *forward price* $F(t,T)$ is the delivery price which would make the contract have zero value at time t. At the time the contract is set up, $t=0$, the forward price therefore equals the delivery price, hence $F(0,T)=K$. The forward prices $F(t,T)$ need not (and will not) necessarily be equal to the delivery price K during the life-time of the contract.
The payoff from a long position in a forward contract on one unit of an asset with price $S(T)$ at the maturity of the contract is

$$S(T)-K.$$

Compared with a call option with the same maturity and strike price K we see that the investor now faces a downside risk, too. He has the obligation to buy the asset for price K.

1.2.2 Futures Markets

Futures prices, like spot prices, are determined on the floor of the exchange by supply and demand, and are quoted in the financial press. *Futures contracts*, however – contracts on assets traded in the futures markets – have various special characteristics. Parties to futures contracts are subject to a daily settlement procedure known as *marking to market*. The initial deposit, paid when the contract is entered into, is adjusted daily by *margin payments* reflecting the daily movement in futures prices. The underlying asset and price are specified in the contract, as is the delivery date. Futures contracts are highly liquid – and indeed, are intended more for trading than for delivery. Being assets, futures contracts may be the subject of *futures options*.
We shall as before write $t=0$ for the time when a contract, or option, is written, t for the present time, T for the expiry time of the option, and T^* for the delivery time specified in the futures (or forward) contract. We will have $T^* \geq T$, and in general $T^* > T$; beyond this, T^* will not affect the pricing of options with expiry T.

Swaps

A *swap* is an agreement whereby two parties undertake to exchange, at known dates in the future, various financial assets (or cash flows) according to a prearranged formula that depends on the value of one or more underlying assets. Examples are currency swaps (exchange currencies) and interest-rate swaps (exchange of fixed for floating set of interest payments).

1.3 Basic Pricing Relations

1.3.1 Storage, Inventory and Convenience Yield

The theory of storage aims to explain the differences between spot and Futures (Forward) prices by analyzing why agents hold inventories. Inventories allow to meet unexpected demand, avoid the cost of frequent revisions in the production schedule and eliminate manufacturing disruption. This motivates the concept of convenience yield as a benefit, that accrues to the owner of the

physical commodity but not to the holder of a forward contract. Thus the convenience yield is comparable to the dividend yield for stocks. A modern view is to view storage (inventory) as a timing option, that allows to put the commodity to the market when prices are high and hold it when the prices are low.
We will model the convenience yield y

- expressed as a rate, meaning that the benefit in a monetary amount for the holder of the commodity will be equal to $S(t)ydt$ over the interval $(t, t + dt)$, if $S(t)$ is the spot price at time t;
- defined as the difference between the positive gain attached to the physical commodity minus the cost of storage. Hence the convenience yield may be positive or negative depending on the period, commodity and cost of storage.

In recent literature the convenience yield is often modelled as a random variable, which allows to explain various shapes of forward curves over time.

1.3.2 Futures Prices and Expectation of Future Spot Prices

The rational expectation hypothesis (REH) (mainly used in the context of interest rates) states that the current futures price $f(t,T)$ for a commodity (interest rate) with delivery a time $T > t$ is the best estimator for the price $S(T)$ of the commodity. In mathematical terms

$$f(t,T) = I\!E[S(T)|\mathcal{F}_t]. \tag{1.1}$$

where $\mathcal{F}_t$ represents the information available at time t. The REH has been statistically tested in many studies for a wide range of commodities (resulting most of the time in a rejection).
When equality in (1.1) does not hold futures prices are biased estimators of future spot prices. If

- $>$ holds, then $f(t,T)$ is an up-ward biased estimate, then risk-aversion among market participants is such that buyers are willing to pay more than the expected spot price in order to secure access to the commodity at time T (political unrest);
- $<$ holds, then $f(t,T)$ is an down-ward biased estimate, this may reflect a perception of excess supply in the future.

No general theory for the bias has been developed. It may depend on the specific commodity, the actual forecast of the future spot price by market participant, and on the risk aversion of the participants.

1.3.3 Spot-Forward Relationship in Commodity Markets

Under the no-arbitrage assumption we have

$$F(t,T) = S(t)e^{(r-y)(T-t)} \tag{1.2}$$

where r is the interest rate at time t for maturity T and y is the convenience yield. We start by proving this relationship for stocks as underlying

Non-dividend paying stocks

Consider the portfolio

	t	T
buy stock	$-S(t)$	delivery
borrow to finance	$S(t)$	$-S(t)e^{r(T-t)}$
sell forward on S		F(t,T)

All quantities are known at t, the time t cash-flow is zero, so the cash-flow at T needs to be zero so we have

$$F(t,T) = S(t)e^{r(T-t)} \tag{1.3}$$

dividend paying stocks

Assume a continuous dividend κ, so we have a dividend rate of $\kappa S(t)dt$, which is immediately reinvested in the stock. We thus have a growth rate of $e^{g(T-t)}$ over the period of the quantity of stocks detained. Thus we only have to buy $e^{-g(T-t)}$ shares of stock S at time time. Replace in the above portfolio and obtain

$$F(t,T) = S(t)e^{(r-g)(T-t)} \tag{1.4}$$

storable commodity

Here the convenience yield plays the role of the dividend an we obtain (1.2).
In case of a linear rate this relationship is of the form

$$F(t,T) = S(t)\left\{1 + (r-y)(T-t)\right\}.$$

With the decomposition $y = y_1 - c$ with y_1 the *benefit from the physical commodity* and c the *storage cost* we have

$$F(t,T) = S(t)\left\{1 + r(T-t) + c(T-t) - y_1(T-t)\right\}.$$

Observe that (1.2) implies

(i) spot and forward are redundant (one can replace the other) and form a linear relationship (unlike options)

(ii) with two forward prices we can derive the value of $S(t)$ and y

(iii) knowledge of $S(t)$ and y allows us to construct the whole forward curve

(iv) for $r - y < 0$ we have backwardation; for $y - r > 0$ we have contango.

1.3.4 Futures Pricing Relations

We start by discussing the subtle but important issue of the difference of the price of a Futures contract i.e. at which we can buy or sell the contract today (for payment at maturity) and the value of a position build in the past and containing this contract.
So consider a Futures contract with a fixed maturity T and a designated underlying.
The price $f(0,T)$ of this contract is defined as the Euro amount the buyer of the contract agrees to pay at date T in order to take delivery of the underlying at date T.
A day later (at t_1) the price of the same contract is $f(t_1,T)$ and may (and will) be different from $f(0,T)$.
So the buyer (the long position) is facing a loss/gain equal to

$$f(t_1,T) - f(0,T)$$

and needs to pay a margin call equal to this amount to the clearing house (Futures exchange). Assuming the position is not closed until maturity T we get

$$f(T,T) - f(0,T) = \underbrace{f(T,T) - f(t_{n-1},T)}_{\text{last day}} + \ldots + \underbrace{f(t_1,T) - f(0,T)}_{\text{first day}}.$$

So the left-hand side represents the profit and loss of a long position P in the Futures contract initiated at time 0. Denoting by $V_p(t)$ the market value of this position at any date t between 0 and T, we know $V_p(0) = 0$ (since this is how the contract is priced).
Also by the convergence assumption $f(T,T) = S(T)$ since it is equivalent to buy a commodity on the spot market and a Futures contract that matures immediately.

In order to find the value $V_p(t)$ of a portfolio containing a Futures contract purchased at date $t = 0$ for delivery at T consider the portfolio consisting of P and a short position P' in a Futures contract entered in at time t.
Payoff at T is

$$\begin{array}{lcll} P & : & -f(0,T) & \text{(buy commodity)} \\ P' & : & f(t,T) & \text{(sell commodity)} \\ P'' = P + P' & : & f(t,T) - f(0,T) & \end{array} .$$

So $V_{P+P'}(T) = f(t,T) - f(0,t)$ and P'' is riskless at time t since the value of all cash flows is known, so by no arbitrage

$$V_{P''}(t) = e^{r(T-t)}(f(t,T) - f(0,T)).$$

Since the value $V_{P'}(t)$ is zero (recall no payment needed to enter a Futures contact) we have

$$V_P(t) = V_{P''}(t) - V_{P'}(t) = e^{-r(T-t)}(f(t,T) - f(0,T)).$$

So the value of a futures contract entered in at 0 at time t is

$$V_P(t) = e^{-r(T-t)}(f(t,T) - f(0,T)) \tag{1.5}$$

Despite their fundamental differences, futures prices $f(t,T)$ and the corresponding forward prices $F(t,T)$, are closely linked. We use the notation $p(t,T)$ for the bond price process.

Proposition 1.3.1. *If the bond price process $p(t,T)$ is predictable, the combined spot-futures market is arbitrage-free if and only if the futures and forward prices agree: for every underlying S and every $t \leq T$,*

$$f_S(t,T) = F_S(t,T).$$

In the important special case of the futures analogue of the Black-Scholes model the bond price process – or interest-rates process – is deterministic, so predictable. We thus only consider the case of deterministic interest rates and a non-dividend paying stock as underlying.
Observe:

(i) Under deterministic or stochastic interest rates the spot-forward relationship is

$$F(t,T) = \frac{S(t)}{p(t,T)}$$

with $p(t,T)$ the price at date t of a zero-coupon bound.

(ii) consider the following sequence of investments in the period $[t,T]$ with subperiods $t, t+1, \ldots, T$

at t: take a long position in $1/p(t,t+1)$ Futures contracts with maturity T

at $t+1$: close this position and invest the proceeds $\frac{1}{p(t,t+1)}\{f(t+1,T) - f(t,T)\}$ on a daily basis until date T with final wealth

$$\frac{1}{p(t,t+1)}\{f(t+1,T) - f(t,T)\}\frac{1}{p(t+1,t+2)\cdot\ldots\cdot p(T-1,T)}.$$

Also take a long position in $\frac{1}{p(t,t+1)p(t+1,t+2)}$ Futures contracts with maturity T.

at $t+2$: close/open positions as above.

at date T: we have the aggregate position

$$\frac{1}{p(t,t+1)\ldots p(T-1,T)}\{f(T;T) - f(t-1,T) + \ldots + f(t+1,T) - f(t,T)\}$$
$$= \frac{f(T,T) - f(t,T)}{p(t,t+1)\cdot\ldots\cdot p(T-1,T)}.$$

Lastly add a position of an investment of $f(t,T)$ Euros in a roll-over lending up to time T, which provides a wealth at T of

$$\frac{1}{p(t,t+1)\cdot\ldots\cdot p(T-1,T)}f(t,T).$$

By addition the portfolio value is

$$\frac{1}{p(t,t+1)\cdot\ldots\cdot p(T-1,T)} f(T,T)$$

and required an initial wealth of $f(t,T)$ since no payments are needed to enter a Futures contract.

In case of deterministic interest rates we find from the no-arbitrage condition

$$p(t,t+1)\cdot\ldots\cdot p(T-1,T) = p(t,t).$$

From $f(T,T) = S(T)$ the final value of the portfolio can thus be written as $\frac{S(T)}{p(t,T)}$ and required an investment of $f(t,T)$.

(iii) The position of buying at t $\frac{1}{p(t,T)}$ shares and keeping them until T requires an investment of $S(t)/p(t,T)$ and has a terminal value of $\frac{S(T)}{p(t,T)}$.

So (ii) and (iii) yield portfolios with same value at date T in all states of the world. By no-arbitrage (observe no in/out-flow of funds) they have the same value at any time t, in particular

$$f(t,T) = \frac{S(t)}{p(t,T)} = F(t,T).$$

1.4 Pricing Formulae for Options

1.4.1 Black-Scholes Formula

The Model

Recall the classical Black-Scholes model

$$\begin{array}{rcll} dB(t) &=& rB(t)dt, & B(0)=1, \\ dS(t) &=& S(t)\left(bdt+\sigma dW(t)\right), & S(0)=p\in(0,\infty), \end{array}$$

with constant coefficients $b \in I\!\!R,\ r,\sigma \in I\!\!R_+$. We write as usual $\tilde{S}(t) = S(t)/B(t)$ for the discounted stock price process (with the bank account being the natural numéraire), and get from Itô's formula

$$d\tilde{S}(t) = \tilde{S}(t)\left\{(b-r)dt + \sigma dW(t)\right\}.$$

Equivalent Martingale Measure

Because we use the Brownian filtration any pair of equivalent probability measures $I\!\!P \sim \mathbb{Q}$ on $\mathcal{F}_T$ is a Girsanov pair, i.e.

$$\left.\frac{d\mathbb{Q}}{dI\!\!P}\right|_{\mathcal{F}_t} = L(t)$$

with

$$L(t) = \exp\left\{-\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma(s)^2 ds\right\},$$

and $(\gamma(t): 0\le t\le T)$ a measurable, adapted d-dimensional process with $\int_0^T \gamma(t)^2 dt < \infty \quad a.s.$. By Girsanov's theorem **??** we have

$$dW(t) = d\tilde{W}(t) - \gamma(t)dt,$$

where $\tilde{W}$ is a $\mathbb{Q}$-Wiener process. Thus the $\mathbb{Q}$-dynamics for $\tilde{S}$ are

$$d\tilde{S}(t) = \tilde{S}(t)\left\{(b - r - \sigma\gamma(t))dt + \sigma d\tilde{W}(t)\right\}.$$

Since $\tilde{S}$ has to be a martingale under $\mathbb{Q}$ we must have

$$b - r - \sigma\gamma(t) = 0 \quad t \in [0, T],$$

and so we must choose

$$\gamma(t) \equiv \gamma = \frac{b - r}{\sigma},$$

(the 'market price of risk'). Indeed, this argument leads to a unique martingale measure, and we will make use of this fact later on. Using the product rule, we find the $\mathbb{Q}$-dynamics of S as

$$dS(t) = S(t)\left\{rdt + \sigma d\tilde{W}\right\}.$$

We see that the appreciation rate b is replaced by the interest rate r, hence the terminology risk-neutral (or yield-equating) martingale measure.
We also know that we have a unique martingale measure $\mathbb{P}^*$ (recall $\gamma = (b - r)/\sigma$ in Girsanov's transformation).

Pricing and Hedging Contingent Claims

Recall that a contingent claim X is a $\mathcal{F}_T$-measurable random variable such that $X/B(T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}^*)$. (We write $\mathbb{E}^*$ for $\mathbb{E}_{\mathbb{P}^*}$ in this section.) By the risk-neutral valuation principle the price of a contingent claim X is given by

$$\Pi_X(t) = e^{\{-r(T-t)\}}\mathbb{E}^*\left[X|\, \mathcal{F}_t\right],$$

with $\mathbb{E}^*$ given via the Girsanov density

$$L(t) = \exp\left\{-\left(\frac{b-r}{\sigma}\right)W(t) - \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 t\right\}.$$

Now consider a European call with strike K and maturity T on the stock S (so $\Phi(T) = (S(T) - K)^+$), we can evaluate the above expected value (which is easier than solving the Black-Scholes partial differential equation) and obtain:

Proposition 1.4.1 (Black-Scholes Formula)**.** *The Black-Scholes price process of a European call is given by*

$$C(t) = S(t)\Phi(d_1(S(t), T - t)) - Ke^{-r(T-t)}\Phi(d_2(S(t), T - t)). \tag{1.6}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s,t) &= d_1(s,t) - \sigma\sqrt{t} = \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \end{aligned}$$

We can also use an arbitrage approach to derive the Black-Scholes formula. For this consider a self-financing portfolio which has dynamics

$$dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)dS(t) = (\varphi_0(t)rB(t) + \varphi_1(t)\mu S(t))dt + \varphi_1(t)\sigma S(t)dW(t).$$

Assume that the portfolio value can be written as

$$V_\varphi(t) = V(t) = f(t, S(t))$$

for a suitable function $f \in C^{1,2}$. Then by Itô's formula

$$dV(t) = (f_t(t, S_t) + f_x(t, S_t)S_t\mu + \frac{1}{2}S_t^2\sigma^2 f_{xx}(t, S_t))dt + f_x(t, S_t)\sigma S_t dW_t.$$

Now we match the coefficients and find

$$\varphi_1(t) = f_x(t, S_t)$$

and

$$\varphi_0(t) = \frac{1}{rB(t)}(f_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(t, S_t)).$$

Then looking at the total portfolio value we find that $f(t, x)$ must satisfy the Black-Scholes partial differential equation

$$f_t(t, x) + rxf_x(t, x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t, x) - rf(t, x) = 0 \tag{1.7}$$

and initial condition $f(T, x) = (x - K)^+$.

1.4.2 Options on Dividend-paying Stocks

We assume that the stock pays a dividend at some fixed rate κ and that the dividend payments are used in full for reinvestment. Consequently, a trading strategy $\varphi = (\varphi_0, \varphi_1)$ is self-financing if its wealth process

$$V_\varphi(t) = \varphi_0(t)B(t) + \varphi_1(t)S(t)$$

satisfies

$$dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)dS(t) + \kappa\varphi_1(t)S(t)dt,$$

or equivalently (using the stochastic dynamics of the stock),

$$dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)(\kappa + \mu)S(t)dt + \varphi_1(t)S(t)\sigma dW(t).$$

Consider now the auxiliary process

$$S^*(t) = e^{\kappa t}S(t).$$

From an application of Itô's lemma we see

$$dS^*(t) = \mu_\kappa S^*(t)dt + \sigma S^*(t)dW(t), \quad \text{with} \quad \mu_\kappa := \mu + \kappa.$$

In terms of this process we have

$$V_\varphi(t) = \varphi_0(t)B(t) + \varphi_1(t)e^{-\kappa t}S^*(t) \ \text{ resp. } \ dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)e^{-\kappa t}dS^*(t).$$

For the discounted wealth $\tilde{V}_\varphi(t) = V_\varphi(t)/B(t)$ we find

$$d\tilde{V}_\varphi(t) = \varphi_1(t)e^{-\kappa t}d\tilde{S}(t) \ \text{ with } \ \tilde{S}_\varphi(t) = S^*/B(t).$$

or

$$d\tilde{V}_\varphi(t) = \varphi_1(t)\sigma\tilde{S}(t)\left(dW(t) + \sigma^{-1}(\mu_\kappa - r)dt\right).$$

Thus we obtain a unique martingale measure $I\!P^*$ by using Girsanov's theorem with $\gamma = \sigma^{-1}(\mu_\kappa - r)$. The dynamics of $\tilde{V}_\varphi(t)$ and $\tilde{S}^*(t)$ are

$$d\tilde{V}_\varphi(t) = \sigma\varphi_1(t)\tilde{S}^*(t)d\tilde{W}(t) \ \text{ and } \ d\tilde{S}^*(t) = \sigma\tilde{S}^*(t)d\tilde{W}(t)$$

with $\tilde{W}(t) = W(t) - (r - \mu_k)\sigma^{-1}t$. We can now simply repeat the argument used to obtain the Black-Scholes formula to prove

Proposition 1.4.2. *The arbitrage price at $t < T$ of a European call on a stock paying dividends at a constant rate κ during the option's lifetime is is given by the risk-neutral valuation formula*

$$C^\kappa(t) = B_t \mathbb{E}^* \left[B_T^{-1}(S_T - K)^+ \middle| \mathcal{F}_t \right] \tag{1.8}$$

or explicitly

$$C^\kappa(t) = \bar{S}(t)\Phi(d_1(\bar{S}(t), T-t)) - Ke^{-r(T-t)}\Phi(d_2(\bar{S}(t), T-t)). \tag{1.9}$$

where $\bar{S}(t) = S(t)e^{-\kappa(t-t)}$ and the functions $d_1(s,t)$ and $d_2(s,t)$ are as above.

Proof. The first equality is the risk-neutral valuation formula. For the second observe the

$$C^\kappa(t) = e^{-r(T-t)}\mathbb{E}^* \left[(S_T - K)^+ \middle| \mathcal{F}_t \right] = e^{-\kappa T} e^{-r(T-t)} \mathbb{E}^* \left[(S_T^* - e^{\kappa T} K)^+ \middle| \mathcal{F}_t \right].$$

The last expectation can now be evaluated similar to the corresponding expectation leading to the Black-Scholes equation.

1.4.3 Black's Futures Options Formula

We turn now to the problem of extending our option pricing theory from spot markets to futures markets. We assume that the stock-price dynamics S are given by geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

and that interest rates are deterministic. We know that there exists a unique equivalent martingale measure, $\mathbb{P}^*$ (for the discounted stock price processes), with expectation $\mathbb{E}^*$. Write

$$f(t) := f_S(t, T^*)$$

for the futures price $f(t)$ corresponding to the spot price $S(t)$. Then risk-neutral valuation gives

$$f(t) = \mathbb{E}^*(S(T^*)|\mathcal{F}_t) \quad (t \in [0, T^*]),$$

while forward prices are given in terms of bond prices by

$$F(t) = S(t)/B(t, T^*) \quad (t \in [0, T^*]).$$

So by Proposition 1.3.1

$$f(t) = F(t) = S(t)e^{r(T^*-t)} \quad (t \in [0, T^*]).$$

So we can use the product rule to determine the dynamics of the futures price

$$df(t) = (\mu - r)f(t)dt + \sigma f(t)dW(t), \quad f(0) = S(0)e^{rT^*}.$$

In the following we study a general Futures market and assume

$$df(t) = \mu f(t)dt + \sigma f(t)dW(t).$$

Again, we say that a futures strategy is self-financing if

$$dV_\varphi^f(t) = \varphi_0(t)dB(t) + \varphi_1(t)df(t).$$

But observe that

$$V_\varphi^f(t) = \varphi_0(t)B(t),$$

since it costs nothing to enter a Futures position.
We call a probability measure $\mathbb{P}^* \sim \mathbb{P}$ a Futures martingale measure, if

$$\tilde{V}_\varphi^f(t) = \frac{V_\varphi^f(t)}{B(t)},$$

follows a (local) martingale.

Lemma 1.4.1. *$\mathbb{P}^* \sim \mathbb{P}$ is a Futures martingale measure if and only if f is a (local) martingale under $\mathbb{P}^*$.*

Proof. Using the product rule we see that $\tilde{V}_\varphi^f(t)$ satisfies for any self-financing φ

$$\begin{aligned} d\tilde{V}_\varphi^f(t) &= B(t)^{-1}\left(\varphi_0(t)dB(t) + \varphi_1(t)df(t)\right) - rB(t)^{-1}\tilde{V}_\varphi^f(t)dt \\ &= B(t)^{-1}\left(\varphi_0(t)dB(t) - r\tilde{V}_\varphi^f(t)dt\right) + B(t)^{-1}\varphi_1(t)df(t) \\ &= B(t)^{-1}\left(V_\varphi^f(t)B(t)^{-1}rB(t)dt - r\tilde{V}_\varphi^f(t)dt\right) + B(t)^{-1}\varphi_1(t)df(t). \blacksquare \end{aligned}$$

As usual we obtain from Girsanov's theorem

Proposition 1.4.3. *The unique martingale measure $\mathbb{P}^*$ on $(\Omega, \mathbb{F})$ for the process f is given by*

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{-\frac{\mu}{\sigma}W(t) - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right\}.$$

Thus the $\mathbb{P}^*$-dynamics for the Futures price f are

$$df(t) = \sigma f(t)d\tilde{W}(t)$$

and the process

$$\tilde{W}(t) = W(t) + \mu\sigma^{-1}t$$

is a $\mathbb{P}^*$-Wiener process. Also

$$f(t) = f_0 \exp\left\{\sigma\tilde{W}(t) - \frac{1}{2}\sigma^2 t\right\}.$$

We turn now to the futures analogue of the Black-Scholes formula, due to Black (1976). We and use the same notation - strike K, expiry T as in the spot case, and write Φ for the standard normal distribution function.

Theorem 1.4.1. *The arbitrage price C of a European futures call option is*

$$C(t) = c(f(t), T-t),$$

where $c(f,t)$ is given by Black's futures options formula:

$$c(f,t) := e^{-rt}(f\Phi(\tilde{d}_1(f,t)) - K\Phi(\tilde{d}_2(f,t))),$$

where

$$\tilde{d}_{1,2}(f,t) := \frac{\log(f/K) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}.$$

Proof. By risk-neutral valuation,

$$C(t) = B(t)\mathbb{E}^*\left[(f(T)-K)^+/B(T)|\mathcal{F}_t\right],$$

with $B(t) = e^{rt}$. For simplicity, we work with $t = 0$; the extension to the general case is immediate. Thus

$$\begin{aligned} C(0) &= \mathbb{E}^*\left[(f(T)-K)^+/B(T)\right] \\ &= \mathbb{E}^*\left[e^{-rT}f(T)\mathbf{1}_D\right] - \mathbb{E}^*\left[e^{-rT}K\mathbf{1}_D\right] \\ &= \mathbf{1}_1 - \mathbf{1}_2 \end{aligned}$$

say, where

$$D := \{f(T) > K\}.$$

Thus

$$\begin{aligned}
\mathbf{1}_2 &= e^{-rT} K \mathbb{P}^*(f(T) > K) \\
&= e^{-rT} K \mathbb{P}^* \left(f(0) \exp\left\{ \sigma \tilde{W}(T) - \frac{1}{2}\sigma^2 T \right\} > K \right),
\end{aligned}$$

where $\tilde{W}$ is a standard Brownian motion under $\mathbb{P}^*$. Now $\xi := -\tilde{W}(T)/\sqrt{T}$ is standard normal, with law Φ under $\mathbb{P}^*$, so

$$\begin{aligned}
\mathbf{1}_2 &= e^{-rT} K \mathbb{P}^* \left(\xi < \frac{\log(f(0)/K) - \frac{1}{2}\sigma^2}{\sigma\sqrt{T}} \right) \\
&= e^{-rT} K \Phi \left(\frac{\log(f(0)/K) - \frac{1}{2}\sigma^2}{\sigma\sqrt{T}} \right) \\
&= e^{-rT} K \tilde{d}_2(f(0), T).
\end{aligned}$$

To evaluate $\mathbf{1}_1$ define an auxiliary probability measure $\hat{\mathbb{P}}$ by setting

$$\left. \frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} \right|_{\mathcal{F}_t} = \exp\left\{ \sigma \tilde{W}(t) - \frac{1}{2}\sigma^2 t \right\},$$

and thus

$$\mathbf{1}_1 = \mathbb{E}^* \left[e^{-rT} f(T) \mathbf{1}_D \right] = e^{-rT} f(0) \hat{\mathbb{P}}(f(T) > K).$$

Since

$$\hat{W}(t) = \tilde{W}(t) - \sigma t$$

is a standard $\hat{\mathbb{P}}$-Wiener process we see

$$f(t) = f_0 \exp\left\{ \sigma \hat{W}(t) - \frac{1}{2}\sigma^2 t \right\}.$$

Thus

$$\begin{aligned}
\mathbf{1}_1 &= e^{-rT} f(0) \hat{\mathbb{P}} \left(f(T) > K \right) \\
&= e^{-rT} f(0) \hat{\mathbb{P}} \left(f_0 \exp\left\{ \sigma \hat{W}(T) - \frac{1}{2}\sigma^2 T \right\} > K \right) \\
&= e^{-rT} f(0) \hat{\mathbb{P}} \left(-\sigma \hat{W}(T) < \log(f(0)/K) + \frac{1}{2}\sigma^2 T \right) \\
&= e^{-rT} f(0) \Phi \left(\tilde{d}_1(f(0), T) \right).
\end{aligned}$$

■

Observe that the quantities $\tilde{d}_1$ and $\tilde{d}_2$ do not depend on the interest rate r. This is intuitively clear from the classical Black approach: one sets up a replicating risk-free portfolio consisting of a position in futures options and an offsetting position in the underlying futures contract. The portfolio requires no initial investment and therefore should not earn any interest.

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