# Lévy copulas: dynamics and transforms of Upsilon-type

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**Abstract**

Lévy processes and infinitely divisible distributions are increasingly defined in terms of their Lévy measure. In order to describe the dependence structure of a multivariate Lévy measure, Tankov (2003) introduced positive Lévy copulas. Together with the marginal Lévy measures they completely describe multivariate Lévy measures on $\mathbb{R}_+^m$. In this paper, we show that any such Lévy copula defines itself a Lévy measure with 1-stable margins, in a canonical way. A limit theorem is obtained, characterising convergence of Lévy measures with the aid of Lévy copulas. Homogeneous Lévy copulas are considered in detail. They correspond to Lévy processes which have a time-constant Lévy copula, and a complete description of homogeneous Lévy copulas is obtained. A general sheme to construct multivariate distributions having special properties is outlined, for distributions with prescribed margins having the same properties. This makes use of Lévy copulas and of certain mappings of Upsilon type. The construction is then exemplified for distributions in the Goldie-Steutel-Bondesson class, the Thorin class and for selfdecomposable distributions.

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# 1 Introduction

The concept of copulas for multivariate probability distributions (or distributional copulas, for short) has an analogue for multivariate Lévy measures, called *Lévy copulas*. The latter concept was introduced in a paper by Tankov [19] for Lévy measures on $\mathbb{R}_+^m$, and extended to Lévy measures on $\mathbb{R}^m$ by Kallsen and Tankov [13], see also the book by Cont and Tankov [10]. Similar to copulas, a Lévy copula describes the dependence structure of a multivariate Lévy measure. The Lévy measure is then completely characterised by knowledge of the Lévy copula and the margins. Here and henceforth, by the *margins* of an $m$-dimensional Lévy measure $\nu$ (or distribution $\mu$) we will always mean the $m$ one-dimensional margins, which are obtained as projections of $\nu$ (or $\mu$) onto the coordinate axes.

An advantage of modelling dependence via Lévy copulas is that the resulting probability law is automatically infinitely divisible. From the applied point of view, the usefulness of modelling Lévy measures hinges to a considerable extent on how feasible it is to obtain insight into relevant properties of the corresponding probability distributions. Much theoretical information in this regard can be gleaned from the book by Sato [16], while numerically there are now powerful methods that in many cases allow rather easy simulation of a probability law from its Lévy measure. In this latter respect, see Cont and Tankov [10] and references given there, cf. also Rosinski [15].

The present paper discusses several aspects of the Lévy copula concept. Recall that a *Lévy measure* is a measure $\nu$ on $\mathbb{R}^m$ which has no atom at zero and satisfies $\int_{\mathbb{R}^m}(|x|^2 \wedge 1)\nu(dx) < \infty$, where $|x| = (x_1^2 + \ldots + x_m^2)^{1/2}$ denotes the Euclidean norm of $x = (x_1, \ldots, x_m)$. We call a Lévy measure *positive* if its support is contained in $\mathbb{R}_+^m = [0, \infty)^m$. For simplicity we shall restrict attention to the class $\mathcal{L}_+^m$ of positive Lévy measures and hence to Lévy copulas living on $[0, \infty]^m$. For the definition of Lévy copulas, we follow the exposition given in Section 5.5 in the book of Cont and Tankov [10]. Accordingly, for every Lévy measure $\nu \in \mathcal{L}_+^m$ the *tail integral* $U = U_\nu$ can be defined as the function $U : [0,\infty]^m \to [0,\infty]$ given by

$$U(x_1,\dots,x_m) := \begin{cases} \nu([x_1,\infty]\times\ldots\times[x_m,\infty]), & (x_1,\dots,x_m) \neq \{0,\dots,0\} \\ \infty, & (x_1,\dots,x_m) = (0,\dots,0). \end{cases}$$

Note that $U(0,\dots,0) = \nu([0,\infty]^m)$ if and only if $\nu$ is infinite. It is convenient when working with Lévy copulas to define $U(0,\dots,0) := \infty$ even for finite Lévy measures as above. This does not alter anything, since $\nu$ is completely described by knowledge of $U$ on $[0,\infty]^m \setminus \{0,\dots,0\}$.

For $\nu \in \mathcal{L}_+^m$, denote the (one-dimensional) margins of $\nu$ by $\nu_1,\dots,\nu_m$. These margins are one-dimensional Lévy measures. In fact, $\nu_1,\dots,\nu_m$ are the Lévy measures of the one-dimensional margins of the probability measure corresponding to $\nu$. To each of them we can associate the tail integral $U_k(x_k)$. Then $U_k(x_k) = U(0,\dots,0,x_k,0,\dots,0)$ for any $x_k \in [0,\infty]$ and we refer to $U_k$ $(k = 1,\dots,m)$ as the *marginal tail integrals* of $\nu$.

In analogy to distributional copulas, Tankov [19] and Cont and Tankov [10] define a *(positive) Lévy copula* to be a function $C : [0,\infty]^m \to [0,\infty]$ such that $C(x_1,\dots,x_m) = 0$ if at least one of the $x_i$ is zero (*groundedness*) and

$$C(\infty,\dots,\infty,x_k,\infty,\dots,\infty) = x_k \quad \forall\, x_k \in [0,\infty], \quad k = 1,\dots,m, \tag{1.1}$$

and such that $C$ is an *m-increasing function*, i.e. $C(x_1,\dots,x_m) \neq \infty$ if $x_1,\dots,x_m$ are not all $\infty$, and for any set $B$ of the form $B = (a_1,b_1]\times\ldots\times(a_m,b_m]$ with $0 \le a_k < b_k \le \infty$ it holds that $\sum \operatorname{sgn}(c)\, C(c) \ge 0$, where the sum is taken over all vertices $c = (c_1,\dots,c_m)$ of $B$, and $\operatorname{sgn}(c)$ is defined as

$$\operatorname{sgn}(c) = \begin{cases} 1, & \text{if } c_k = a_k \text{ for an even number of vertices,} \\ -1, & \text{if } c_k = a_k \text{ for an odd number of vertices.} \end{cases}$$

From this follows easily that if $C$ is a Lévy copula, then

$$\chi_C([0,b_1]\times\ldots\times[0,b_m]) := C(b_1,\dots,b_m), \quad 0 \le b_1,\dots,b_m \le \infty, \tag{1.2}$$

can be extended to a unique (positive) measure $\chi_C$ on the Borel sets of $[0,\infty]^m$ such that $\chi_C$ has no atom at $(\infty,\dots,\infty)$ and has uniform (i.e. standard Lebesgue) margins, i.e.

$$\chi_C([0,\infty]^{k-1}\times[0,x_k]\times[0,\infty]^{m-k}) = x_k, \quad k = 1,\dots,m. \tag{1.3}$$

Conversely, for every measure $\chi$ on $[0,\infty]^m$ with these properties, (1.2) defines a unique Lévy copula.

The most important feature of Lévy copulas is that, analogous to distributional copulas, they allow to separate the margins and the dependence structure of Lévy measures. More precisely, Tankov [19] proved that for any $\nu \in \mathcal{L}_+^m$ with tail integral $U$ and marginal tail integrals $U_1, \ldots, U_m$, there exists a (positive) Lévy copula $C$ such that

$$U(x_1, \ldots, x_m) = C(U_1(x_1), \ldots, U_m(x_m)) \quad \forall \; x_1, \ldots, x_m \in [0, \infty]. \tag{1.4}$$

The Lévy copula $C$ is uniquely determined on $\operatorname{Ran} U_1 \times \ldots \times \operatorname{Ran} U_m$ (where Ran denotes range of a mapping). Conversely, if $C$ is a positive Lévy copula and $U_1, \ldots, U_m$ are tail integrals of one-dimensional positive Lévy measures $\nu_1, \ldots, \nu_m$, then (1.4) defines a Lévy measure $\nu \in \mathcal{L}_+^m$ with tail integral $U$ and marginal Lévy measures $\nu_1, \ldots, \nu_m$. See also Cont and Tankov [10]. We shall refer to any Lévy copula $C$ satisfying (1.4) as a Lévy copula *associated with* $\nu \in \mathcal{L}_+^m$.

The present paper is organised as follows: Section 2 establishes a limit result for sequences of Lévy measures and Lévy copulas: we show that a sequence of Lévy measures converges vaguely to another Lévy measure if and only if the marginal Lévy measures converge vaguely, and the Lévy copulas converge pointwise on a suitable subset of $[0, \infty]^m$.

Section 3 discusses the special class of homogeneous Lévy copulas in more detail. They arise naturally as Lévy copulas which are constant in time for Lévy processes: if $(L^{(t)})_{t \geq 0}$ is a Lévy process with Lévy measure $\nu^{(t)}$ at time $t$ and if the Lévy copula $C^{(1)}$ of $\nu^{(1)}$ is homogeneous, then $C^{(1)}$ is also a Lévy copula for $\nu^{(t)}$ for any $t > 0$. Furthermore, homogeneous Lévy copulas constitute the class of possible limits of Lévy copulas of Lévy processes as time approaches 0 or $\infty$. We further obtain a complete characterisation of homogeneous Lévy copulas and investigate some conditions which must be satisfied for a Lévy measure to have a homogeneuous Lévy copula.

Section 4 is concerned with the construction of Lévy measures and distributions with special structures and prescribed margins. Suppose that $\nu_1, \ldots, \nu_m$ are one-dimensional Lévy measures, all of which have a similar structure, such as being selfdecomposable, say. In Section 4.1 we outline a general scheme how Lévy copulas can be used to construct a Lévy measure $\nu$ with margins $\nu_1, \dots, \nu_m$ and which has the same structure, e.g. selfdecomposability. Apart from Lévy copulas the method requires certain mappings which are of *Upsilon type*. For example, the mapping $\Upsilon_0$, which was introduced by Barndorff-Nielsen and Thorbjørnsen [6, 7] and Barndorff-Nielsen, Maejima and Sato [3] maps the class of infinitely divisible distributions bijectively onto the Goldie-Steutel-Bondesson class. The general construction theme is then exemplified in Sections 4.2 – 4.4 for Lévy measures in the Goldie-Steutel-Bondesson class, for selfdecomposable Lévy measures and for Lévy measures in the Thorin class. Section 4.5 investigates the action of the mapping $\Upsilon_0$ on Lévy copulas in more detail.

In the final section, we show that every Lévy copula $C$ defines itself a Lévy measure $\nu_C$ of infinite variation with one-stable margins in a canonical way. It is then shown that a Lévy copula is homogeneous if and only if $\nu_C$ is 1-stable, thus proving the characterisation of homogeneous Lévy copulas appearing in Section 3.

# 2 Lévy copulas and convergence of Lévy measures

In this section we obtain a limit result for Lévy measures, characterising convergence of a sequence of Lévy measures by convergence of the margins and of the Lévy copulas. Let $\mu$ be an infinitely divisible distribution on $\mathbb{R}^m$ with characteristic triplet $(A, \nu, \gamma)$. Recall that $\nu$ is completely characterised by $(A, \nu, \gamma)$, and that the characteristic function $\widehat{\mu}$ of $\mu$ satisfies

$$\widehat{\mu}(z) = \exp\{-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^m} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{|x|\le 1})\, d\nu(x)\}, \quad z \in \mathbb{R}^m.$$

Here, $A$ is a symmetric nonnegative-definite $m \times m$-matrix, $\nu$ is the Lévy measure of $\mu$, and $\gamma \in \mathbb{R}^m$ is a constant. $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^m$.

Denote by $C_\#$ the class of bounded continuous functions from $\mathbb{R}^m$ to $\mathbb{R}$ vanishing in a neighbourhood of the origin. Let $(\mu^{(n)})_{n\in\mathbb{N}}$ be a sequence of infinitely divisible distributions on $\mathbb{R}^m$ with characteristic triplets $(A^{(n)}, \nu^{(n)},$

$\gamma^{(n)}$). For any $\varepsilon > 0$ define symmetric nonnegative-definite matrices $A^{(n),\varepsilon}$ by

$$\langle z, A^{(n),\varepsilon} z\rangle = \langle z, A^{(n)} z\rangle + \int_{|x|\le\varepsilon} \langle z, x\rangle^2 \, d\nu^{(n)}(x), \quad z \in \mathbb{R}^m.$$

Then it is known that $(\mu^{(n)})_{n\in\mathbb{N}}$ converges weakly to an infinitely divisible distribution $\mu$ with characteristic triplet $(A, \nu, \gamma)$ if and only if

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} f(x)\, d\nu^{(n)}(x) = \int_{\mathbb{R}^m} f(x)\, d\nu(x) \quad \forall\, f \in \mathcal{C}_{\#}, \tag{2.1}$$

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} |\langle z, A^{(n),\varepsilon} z\rangle - \langle z, Az\rangle| = 0 \quad \forall\, z \in \mathbb{R}^m, \tag{2.2}$$

$$\lim_{n\to\infty} \beta^{(n)} = \beta, \tag{2.3}$$

where

$$\beta := \gamma - \int_{|x|\le 1} x|x|^2 \, d\nu(x)$$

and $\beta^{(n)}$ is defined similarly. See e.g. Sato [16], Theorem 8.7. Hence the appropriate convergence concept for Lévy measures is described by relation (2.1). We shall write $\nu^{(n)} \overset{\#}{\to} \nu$ for this type of convergence of Lévy measures. Standard arguments show that for positive Lévy measures $\nu^{(n)}$ and $\nu$, $\nu^{(n)} \overset{\#}{\to} \nu$ as $n \to \infty$ if and only if $\nu^{(n)}$ converges vaguely to $\nu$ on $[0, \infty]^m \setminus \{0, \ldots, 0\}$, which is further equivalent to the pointwise convergence of the corresponding tail integrals $U^{(n)}(x)$ to $U(x)$ at every point $x \in \mathcal{G}_1 \times \ldots \times \mathcal{G}_m$, where

$$\mathcal{G}_i := \{x_i \in (0, \infty] : U_i \text{ continuous in } x_i\} \cup \{0\}, \tag{2.4}$$

and the $U_i$ denote the marginal tail integrals of $\nu$, $i = 1, \ldots, m$. See [2], Lemma 3.2, for a detailed proof of this.

We can now show that a sequence of Lévy measures converges to a Lévy measure if and only if the margins converge and the Lévy copulas converge pointwise on a suitable subset. This is an analogue of a result of Deheuvels [11] (see also Lindner and Szimayer [14]) for distributional copulas.

**Theorem 2.1** *Let $(\nu^{(n)})_{n\in\mathbb{N}} \subset \mathcal{L}_+^m$, $\nu \in \mathcal{L}_+^m$, with margins $\nu_i^{(n)}$ and $\nu_i$ ($i = 1, \ldots, m$), and associated Lévy copulas $C^{(n)}$ and $C$, respectively. Then $\nu^{(n)} \overset{\#}{\to} \nu$ as $n \to \infty$ if and only if $\nu_i^{(n)} \overset{\#}{\to} \nu_i$ as $n \to \infty$ for $i = 1, \ldots, m$, and $C^{(n)}$ converges pointwise to $C$ on $Ran\, U_1 \times \ldots \times Ran\, U_m$ as $n \to \infty$, where the $U_i$*

*denote the marginal tail integrals of* $\nu$*. In that case, the convergence of* $C^{(n)}$ *to* $C$ *is uniform on any set of the form* $(\overline{Ran\, U_1}\times\ldots\times\overline{Ran\, U_m})\cap(K_1\times\ldots\times K_m)$*, where* $K_i$ *is a compact subset of* $[0,\infty)$*, or* $K_i = \{\infty\}$*.*

**Proof.** Since any Lévy copula $D$ defines a measure $\chi_D$ with uniform margins via (1.2), it follows readily that for $k \in \{1,\ldots,m\}$ and $u_1,\ldots,u_k,v_1,\ldots,v_k \in [0,\infty)$ the following Lipschitz condition holds:

$$|D(u_1,\ldots,u_k,\infty,\ldots,\infty) - D(v_1,\ldots,v_k,\infty,\ldots,\infty)| \le \sum_{i=1}^{k} |u_i - v_i|, \quad (2.5)$$

see also Lemma 3.2 in Kallsen and Tankov [13]. Let $\mathcal{M}_i := \{U_i(x_{u,i}) : x_{u,i} \in \mathcal{G}_i\}$, where the $\mathcal{G}_i$ are as in (2.4)

Suppose that $\nu^{(n)} \overset{\#}{\to} \nu$ as $n \to \infty$. Then $\nu_i^{(n)} \overset{\#}{\to} \nu_i$ as $n \to \infty$ for all $i = 1,\ldots,m$ (e.g. using the characterization of vague convergence in terms of the convergence of the tail integrals, as stated before this Theorem). Let $(u_1\ldots,u_m) \in \mathcal{M}_1\times\ldots\times\mathcal{M}_m$ such that $u_i = U_i(x_{u,i})$, $x_{u,i} \in \mathcal{G}_i$, and without loss of generality suppose that $u_1,\ldots,u_k \neq \infty$, $u_{k+1} = \ldots = u_m = \infty$ for some $k \in \{1,\ldots,m\}$. Then (2.5) yields

$$\begin{aligned}
&|C^{(n)}(u_1,\ldots,u_m) - C(u_1,\ldots,u_m)| \\
&\le \left|C^{(n)}(U_1(x_{u,1}),\ldots,U_m(x_{u,m})) - C^{(n)}(U_1^{(n)}(x_{u,1}),\ldots,U_m^{(n)}(x_{u,m}))\right| \\
&\quad + \left|C^{(n)}(U_1^{(n)}(x_{u,1}),\ldots,U_m^{(n)}(x_{u,m})) - C(U_1(x_{u,1},\ldots,U_m(x_{u,m}))\right| \\
&\le \sum_{i=1}^{k} |U_i(x_{u,i}) - U_i^{(n)}(x_{u,i})| \\
&\quad + |U^{(n)}(x_{u,1},\ldots,x_{u,m}) - U(x_{u,1},\ldots,x_{u,m})|.
\end{aligned}$$

The convergence of the tail integrals on $\mathcal{G}_i$, $i = 1,\ldots m$, and on $\mathcal{G}_1\times\ldots\times\mathcal{G}_m$, respectively, then implies convergence of $C^{(n)}$ to $C$ at $(u_1,\ldots,u_m)$. For compact $K_1,\ldots,K_k \subset [0,\infty)$, standard arguments using the equicontinuity of $\{C; C^{(n)} : n \in \mathbb{N}\}$ on $K_1\times\ldots\times K_k\times\{\infty\}\times\ldots\times\{\infty\}$ by (2.5) then imply that the convergence of $C^{(n)}$ to $C$ is actually uniform on $(\mathcal{M}_1\times\ldots\times\mathcal{M}_m)\cap(K_1\times\ldots\times K_k\times\{\infty\}\times\ldots\times\{\infty\})$ and hence on $(\overline{\mathrm{Ran}\, U_1}\times\ldots\times\overline{\mathrm{Ran}\, U_m})\cap(K_1\times\ldots\times K_k\times\{\infty\}\times\ldots\times\{\infty\})$, since $\overline{\mathcal{M}_i} = \overline{\mathrm{Ran}\, U_i}$.

For the converse, suppose that $\nu_i^{(n)} \overset{\#}{\to} \nu_i$ as $n \to \infty$, and that $C^{(n)}$ converges pointwise to $C$ on $\mathcal{M}_1 \times \ldots \times \mathcal{M}_m$. Then for $x = (x_1, \ldots, x_m)$ with $x_i \in \mathcal{G}_i$, $i = 1, \ldots, m$, taken such that $x_1, \ldots, x_k \neq 0$, $x_{k+1} = \ldots = x_m = 0$, $k \in \{1, \ldots, m\}$, the Lipschitz condition (2.5) implies that

$$\begin{aligned}
|U^{(n)}(x) - U(x)| \leq & \\
& \left| C^{(n)}(U_1^{(n)}(x_1), \ldots, U_m^{(n)}(x_m)) - C^{(n)}(U_1(x_1), \ldots, U_m(x_m)) \right| \\
& + \left| C^{(n)}(U_1(x_1), \ldots, U_m(x_m)) - C(U_1(x_1), \ldots, U_m(x_m)) \right| \\
\leq & \sum_{i=1}^{k} |U_i^{(n)}(x_i) - U_i(x_i)| \\
& + |C^{(n)}(U_1(x_1), \ldots, U_m(x_m)) - C(U_1(x_1), \ldots, U_m(x_m))|.
\end{aligned}$$

This gives $\lim_{n\to\infty} U^{(n)}(x) = U(x)$ for $x \in \mathcal{G}_1 \times \ldots \times \mathcal{G}_m$, so that vague convergence of $\nu^{(n)}$ to $\nu$ as $n \to \infty$ follows. ■

Recalling that weak convergence of infinitely divisible distributions can be described by convergence of the characteristic triplets as in (2.1) - (2.3), we obtain the following corollary to Theorem 2.1:

**Corollary 2.2** *Let $(\mu^{(n)})_{n\in\mathbb{N}}$ and $\mu$ be infinitely divisible distributions with characteristic triplets $(A^{(n)}, \nu^{(n)}, \gamma^{(n)})$ and $(A, \nu, \gamma)$, such that $\nu$ and $\nu^{(n)}$ are in $\mathcal{L}_+^m$. Let $\mu^{(n)} = (\mu_1^{(n)}, \ldots, \mu_m^{(n)})$ and $\mu = (\mu_1, \ldots, \mu_m)$. Suppose that $A^{(n)}$ converges pointwise to $A$ as $n \to \infty$. Then $\mu^{(n)}$ converges weakly to $\mu$ as $n \to \infty$ if and only if all the margins $\mu_i^{(n)}$ converge weakly to $\mu_i$ as $n \to \infty$, $i = 1, \ldots, m$, and the Lévy copula of $\nu_n$ converges pointwise to the Lévy copula of $\nu$ on $Ran\, U_1 \times \ldots \times Ran\, U_m$ as $n \to \infty$, where the $U_i$ denote the marginal tail integrals of $\nu$.*

It should be noted that the assumption $\lim_{n\to\infty} A^{(n)} = A$ is somewhat restrictive. It implies that in the limit the Lévy measures do not contribute to an extra Gaussian part. This then makes an easy description by the Lévy copula convergence feasible.

**Proof.** In the following we will refer to (2.1) – (2.3) by $(2.1)_m$ – $(2.3)_m$ and $(2.1)_1$ – $(2.3)_1$, respectively, according to whether we consider distributions on $\mathbb{R}^m$ (such as $\mu^{(n)}$ and $\mu$) or distributions on $\mathbb{R}$ (such as the margins, $\mu_i^{(n)}, \mu_i$).

That weak convergence of $\mu^{(n)}$ to $\mu$ as $n \to \infty$ implies convergence of the margins and of the Lévy copulas is clear by the continuous mapping theorem, $(2.1)_m$ and Theorem 2.1. For the converse, suppose that $\mu_i^{(n)}$ converges weakly to $\mu_i$ as $n \to \infty$ for $i = 1, \ldots, m$, and that the Lévy copulas converge. Then $(2.1)_m$ holds by $(2.1)_1$ and Theorem 2.1. The characteristic triplet of $\mu_i$ is $(A_{ii}, \nu_i, \widetilde{\gamma}_i)$, where $A_{ii}$ denotes the $i$'th diagonal element of $A$, $\widetilde{\gamma}_i = \gamma_i + \int_{\mathbb{R}^m_+} x_i(1_{|x_i|\le 1} - 1_{|x|\le 1})\, d\nu(x)$, and $\gamma_i$ denotes the $i$'th coordinate of $\gamma$, see Sato [16], Proposition 11.10. Let $\widetilde{\beta}_i := \widetilde{\gamma}_i - \int_{|x_i|\le 1} x_i |x_i|^2\, d\nu_i(x_i)$. To show $(2.2)_m$ and $(2.3)_m$, note that convergence of $A^{(n)}$ to $A$ implies convergence of $A_{ii}^{(n)}$ to $A_{ii}$. Since $\mu_i^{(n)}$ converges weakly to $\mu_i$, $(2.2)_1$ and $(2.3)_1$ imply that

$$\lim_{\varepsilon \to 0} \limsup_{n\to\infty} \left| \int_{|x_i|\le\varepsilon} x_i^2\, d\nu_i^{(n)}(x_i) \right| = 0, \tag{2.6}$$

and that $\widetilde{\beta}_i^{(n)}$ converges to $\widetilde{\beta}_i$ as $n \to \infty$. Again, by convergence of $A_n$ to $A$ and (2.6) it then follows, for any $z \in \mathbb{R}^m$, that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} |\langle z, A^{(n),\varepsilon} z\rangle - \langle z, Az\rangle| \\
&\le \ |z|^2 \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \left| \int_{|x|\le\varepsilon} |x|^2\, d\nu^{(n)}(x) \right| \\
&\le \ |z|^2 \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \sum_{i=1}^m \left| \int_{|x_i|\le\varepsilon} x_i^2\, d\nu_i^{(n)}(x_i) \right| = 0.
\end{aligned} \tag{2.7}$$

This shows $(2.2)_m$. For $(2.3)_m$, note that

$$\beta_i - \widetilde{\beta}_i = \int_{|x_i|\le 1} x_i^3\, d\nu_i(x_i) - \int_{|x|\le 1} x_i |x|^2\, d\nu(x) - \int_{\mathbb{R}^m_+} x_i(1_{|x_i|\le 1} - 1_{|x|\le 1})\, d\nu(x),$$

where $\beta_i$ denotes the $i$'th coordinate of $\beta$ (as appearing in $(2.3)_m$). From $\nu^{(n)} \xrightarrow{\#} \nu$, (2.6) and (2.7) one can show that $\beta_i^{(n)} - \widetilde{\beta}_i^{(n)}$ converges to $\beta_i - \widetilde{\beta}_i$ as $n \to \infty$. Since $\widetilde{\beta}_i^{(n)}$ converges to $\widetilde{\beta}_i$, this proves that $\beta_i^{(n)}$ converges to $\beta_i$ as $n \to \infty$, verifying $(2.3)_m$. This finishes the proof. ■

# 3 Homogeneous Lévy copulas

In this section we discuss the special class of homogeneous Lévy copulas. A Lévy copula $C$ is called *homogeneous* (of order 1), if

$$C(u_1, \ldots, u_m) = t\, C(u_1/t, \ldots, u_m/t) \quad \forall\, u_1, \ldots, u_m \in [0, \infty] \quad \forall\, t > 0.$$

Examples of homogeneous Lévy copulas are the Lévy copula of *complete dependence*

$$C(u_1, \ldots, u_m) = \min\{u_1, \ldots, u_m\},$$

the copula of *independence*

$$C(u_1, \ldots, u_m) = \sum_{i=1}^{m} u_i\, 1_{\{u_1 = \ldots = u_{i-1} = u_{i+1} = \ldots = u_m = \infty\}},$$

and the family of *Clayton Lévy copulas*, defined for $\theta > 0$ by

$$C(u_1, \ldots, u_m) = \left( \sum_{i=1}^{m} u_i^{-\theta} \right)^{-1/\theta},$$

$u_1, \ldots, u_m \in [0, \infty]$, see Cont and Tankov [10], Chapter 5. Examples of non-homogeneous Lévy copulas are given in Examples 3.2 and 3.3 below. A complete characterisation of homogeneous Lévy copulas will be given in Theorem 3.4.

Homogeneous Lévy copulas appear naturally, because they correspond to Lévy processes with time constant Lévy copulas: let $(L^{(t)})_{t \geq 0}$ be a Lévy process in $\mathbb{R}^m$. Then at any time $t$, $L^{(t)}$ has an infinitely divisible distribution. If $\nu^{(t)}$ denotes the Lévy measure of $L^{(t)}$ then $\nu^{(t)} = t\nu^{(1)}$. Now suppose that $\nu^{(1)} \in \mathcal{L}_+^m$ with associated Lévy copula $C^{(1)}$. Then it follows readily that

$$C^{(t)}(u_1, \ldots, u_m) := t\, C^{(1)}(u_1/t, \ldots, u_m/t), \quad \forall\, u_1, \ldots, u_m \in [0, \infty], \tag{3.1}$$

gives a Lévy copula associated with $\nu^{(t)}$. In particular, if $C^{(1)}$ is homogeneous, then the Lévy process is described by the same Lévy copula $C^{(1)}$ at any time $t$. On the other hand, if $C$ is a Lévy copula and $(L^{(t)})_{t \geq 0}$ is a Lévy process such that $C$ is associated with $\nu^{(t)}$ for every $t$, and if there is some $\varepsilon > 0$ such that $\operatorname{Ran} U_i^{(1)} \supset [0, \varepsilon]$ for all $i = 1, \ldots, m$ (where $U_i^{(1)}$ denote the marginal volume functions of $\nu^{(1)}$), then $C$ must be homogeneous. This follows from the fact that the Lévy copula of $\nu^{(t)}$ is unique on $t(\operatorname{Ran} U_1^{(1)} \times \ldots \times \operatorname{Ran} U_m^{(1)})$ for any $t > 0$ by Tankov's analogue of Sklar's theorem for Lévy copulas, as stated in the Introduction. This shows that the Lévy copulas at times $t$ and 1 satisfy (3.1), showing that $C$ is homogeneous.

We now turn to convergence of Lévy copulas of Lévy processes as time goes to infinity and to zero. Again, the homogeneous Lévy copulas appear naturally as possible limit copulas.

**Theorem 3.1** *Let $(L^{(t)})_{t\geq 0}$ be a Lévy process with positive Lévy measure and with Lévy copula $C^{(t)}$ at time $t$ given by (3.1). Then:*
*(a) $C^{(t)}$ converges pointwise to a finite function $D$ on $[0,\infty]^m \setminus \{(\infty,\ldots,\infty)\}$ as $t \to \infty$ if and only if all for all directions $(u_1,\ldots,u_m) \in \mathbb{R}_+^m$ the directional derivative of $C^{(1)}$ exists at the origin. In that case, the function $D$ is a homogeneous Lèvy copula. The convergence is uniform on $[0,\infty]^m \setminus \{(\infty,\ldots,\infty)\}$ if and only if $C^{(1)}$ is homogeneous.*
*(b) If $C^{(t)}$ converges pointwise to a finite function $D$ on $[0,\infty]^m \setminus \{(\infty,\ldots,\infty)\}$ as $t \to 0$, then the function $D$ is a homogeneous Lévy copula. $C^{(t)}$ converges uniformly on $[0,\infty]^m \setminus \{(\infty,\ldots,\infty)\}$ to $D$ as $t \to 0$ if and only if $\|C^{(1)} - D\|_\infty < \infty$, where $\|\cdot\|_\infty$ denotes the supremum norm on $[0,\infty]^m \setminus \{\infty,\ldots,\infty\}$.*

**Proof.** From (3.1) follows readily that if $C^{(t)}$ converges pointwise to a finite function $D$ on $[0,\infty]^m \setminus \{\infty,\ldots,\infty\}$ as $t \to \infty$ or $t \to 0$, then $D$ must be a homogeneous Lévy copula. Further, noting that for $u = (u_1,\ldots,u_m)$ and $t > 0$ we have

$$t\, C^{(1)}(u/t) = \frac{C^{(1)}(u/t) - C^{(1)}(0)}{1/t},$$

it follows that $\lim_{t\to\infty} C^{(t)}(u)$ exists if and only if if the directional derivative of $C^{(1)}$ in direction $u$ exists at the origin. If $C^{(1)}$ is homogeneous, then uniform convergence of $C^{(t)}$ as $t \to \infty$ is clear. For the converse, suppose uniform convergence, but that $C^{(1)}$ is not homogeneous. Then there is $u \in [0,\infty]^m$ and $t_0 > 0$ such that $|C^{(1)}(t_0 u) - t_0 C^{(1)}(u)| =: \varepsilon > 0$. From the uniform convergence follows the existence of $t_1 > 0$ such that $|t\, C^{(1)}(v/t) - D(v)| \leq \varepsilon$ forevery $v \in [0,\infty]^m \setminus \{\infty,\ldots,\infty\}$ and every $t > t_1$. Using the homogeneity of $D$ we conclude for $t > t_1$

$$\begin{aligned} |tt_0C^{(1)}(u) - tt_0D(u)| &= \left|t_0tC^{(1)}(tu/t) - t_0D(tu)\right| \le t_0\varepsilon, \\ |tC^{(1)}(t_0u) - tt_0D(u)| &= \left|tC^{(1)}(tt_0u/t) - D(tt_0u)\right| \le \varepsilon. \end{aligned}$$

This implies

$$t\varepsilon = t|t_0C^{(1)}(u) - C^{(1)}(t_0u)| \le (1+t_0)\varepsilon \quad \forall\, t \ge t_1,$$

which clearly is a contradiction. This proves $(a)$.

For the proof of $(b)$, note that, by homogeneity of $D$, $C^{(t)}$ converges uniformly to $D$ as $t \to 0$ if and only if $t|C^{(1)}(v/t) - D(v/t)|$ converges uniformly in $v$ to 0. But this is equivalent to $\|C^{(1)} - D\|_\infty < \infty$. ■

That there are Lévy processes for which the Lévy copulas do not converge as time goes to 0 or to $\infty$ is established in the following example:

**Example 3.2** Let the (distributional) copulas $H_1$ and $H_2$ on $[0,1]^2$ be given by $H_1(u,v) := uv$ and $H_2(u,v) := \min\{u,v\}$. For any integer $n \in \mathbb{Z}$ and $u, v \in [2^n, 2^{n+1}]$ let

$$C^{(1)}(u,v) := 2^n + 2^n H_i\left(\frac{u-2^n}{2^n}, \frac{v-2^n}{2^n}\right),$$

where $i = 1$ if $n$ is odd and $i = 2$ if $n$ is even. If $u \in [2^n, 2^{n+1}]$ for some $n$ and $v > 2^{n+1}$, set $C^{(1)}(u,v) := C^{(1)}(u, 2^{n+1})$, and if $u > v$ set $C^{(1)}(u,v) = C^{(1)}(v,u)$. It can be easily checked that $C^{(1)}$ defines a Lévy copula. Let $u_n := 2^n + 2^{n-1}$. Then $C^{(1)}(u_n, u_n) = u_n$ if $n$ is even, and $C^{(1)}(u_n,u_n) = 2^n + 2^{n-2}$ if $n$ is odd. In particular,

$$\frac{C^{(1)}(u_n,u_n)}{u_n} = \begin{cases} 5/6, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}$$

This shows that for a Lévy process with Lévy copula $C^{(t)}$ at time $t > 0$, $C^{(t)}(1,1)$ does neither converge as $t \to 0$ nor as $t \to \infty$.

That pointwise convergence of Lévy copulas as $t \to \infty$ does not imply uniform convergence can be seen from the following:

**Example 3.3** Consider a Lévy process such that the Lévy copula at time 1 is given by

$$C^{(1)}(u_1,\dots,u_m) := \log\left(\left(\sum_{i=1}^{m}\frac{e^{-u_i}}{1-e^{-u_i}}\right)^{-1}+1\right).$$

This Lévy copula was introduced in Tankov [19], see Cont and Tankov [10], page 150. Let $D_\infty(u_1,\dots,u_m) := \left(\sum_{i=1}^m (1/u_i)\right)^{-1}$ and $D_0(u_1,\dots,u_m) := \min\{u_1,\dots,u_m\}$. Then easy calculations show that $C^{(t)}$ converges pointwise to $D_\infty$ as $t\to\infty$. The convergence is not uniform, since $C^{(1)}$ is not homogeneous. On the other hand, it is easy to show that $\|C^{(1)}-D_0\|_\infty<\infty$, so that $C^{(t)}$ converges uniformly to $D_0$ as $t\to 0$.

There remains the question whether there are Lévy processes such that the Lévy copula $C^{(t)}$ converges pointwise but not uniformly as $t\to 0$. By now we have not been able to decide this question.

The next theorem characterises homogeneous Lévy copulas. We denote $S := \{\xi\in\mathbb{R}^m : |\xi| = 1\}$ and

$$S_+ := \{(\xi_1,\dots,\xi_m)\in S : \xi_1\ge 0,\dots,\xi_m\ge 0\}.$$

**Theorem 3.4** *A function $C : [0,\infty)^m \to [0,\infty]$ is a homogeneous Lévy copula if and only if there exists a finite (positive) measure $\lambda$ on $S_+$ such that with $\xi = (\xi_1,\dots,\xi_m)$*

$$\int_{S_+} \xi_i\,\lambda(d\xi) = 1 \quad \forall\, i=1,\dots,m, \tag{3.2}$$

*and that $C$ has the representation*

$$C(b_1,\dots,b_m) = \int_{S_+} \min\left(b_1\xi_1,\dots,b_m\xi_m\right)\,\lambda(d\xi) \quad \forall\, b_1,\dots,b_m\in[0,\infty]^m \tag{3.3}$$

*with the convention $b_i\xi_i := \infty$ for $b_i=\infty$. A Lévy copula $C$ is homogeneous if and only if there exists a finite measure $\lambda$ on $S_+$ such that* (3.3) *holds.*

The proof relies on showing that the measure $\chi_C$ defined in (1.2) can be transformed to the Lévy measure of a 1-stable distribution, under a suitable inversion map. We will investigate this mapping in more detail in Section 5 and give a proof of Theorem 3.4 there.

We conclude this section by showing that homogeneous Lévy copulas are rarely associated with finite Lévy measures that have no mass on the axes:

**Theorem 3.5** *Let $\nu$ be a finite Lévy measure, concentrated on $(0,\infty)^m$, and suppose that the Lévy copula $C$ associated with $\nu$ is homogeneous. Then $C$ must be the Lévy copula of complete dependence, i.e.*

$$C(u_1,\dots,u_m) = \min\{u_1,\dots,u_m\} \quad \forall\, u_1,\dots,u_m \in [0,\infty].$$

**Proof.** Denote by $M$ the total mass of $\nu$ and its (marginal) tail integrals by $U_i$ and $U$. Then $\lim_{x_i\to 0} U_i(x_i) = M$ for $i \in \{1,\dots,m\}$, and $\lim_{x\to 0} U(x,\dots,x) = M$. By (1.4), $C(U_1(x),\dots,U_m(x))$ converges to $M$ as $x \to 0$. From the continuity property (2.5) then follows $C(M,\dots,M) = M$. Since $C$ was assumed to be homogeneous, we conclude $C(u,\dots,u) = u$ for any $u > 0$. Now let $u_1,\dots,u_m \in [0,\infty]$ and suppose w.l.o.g. that their minimum is at $u_1$. Then

$$u_1 = C(u_1,\dots,u_1) \le C(u_1,u_2,\dots,u_m) \le C(u_1,\infty,\dots,\infty) = u_1,$$

showing the claim. ■

Note that Theorem 3.5 does not contradict the fact that all one-dimensional Lèvy measures can be coupled with every homogeneous Lévy copula via (1.4) to give a Lèvy measure on $[0,\infty)^m$. But it states that the resulting Lévy measure $\nu$ must necessarily satisfy $\nu([0,\infty)^m \setminus (0,\infty)^m) > 0$, if the homogeneous Lévy copula is not the Lévy copula of complete dependence.

# 4 Copulas and transformations of Upsilon type

In general, a model constructed from a set of infinitely divisible one-dimensional marginals and a chosen Lévy copula is not guaranteed to have useful properties beyond the infinite divisibility. This is true even if the marginals have special properties. For instance, selfdecomposability of all the one-dimensional marginals does not imply selfdecomposability of the constructed $m$-dimensional model, as can be seen from the following example. We say that a Lévy measure is stable or self-decomposable, if it is the Lévy measure of a stable or self-decomposable infinitely divisible distribution, respectively. For the definitions and properties of such distributions, we refer to Sato [16], Chapters 13-15.

**Example 4.1** Let $\nu_1$ and $\nu_2$ be one-dimensional positive Lévy measures with stable margins such that $U_1(x_1) = x_1^{-\alpha}$ and $U_2(x_2) = x_2^{-\beta}$, where $0 < \alpha, \beta < 2$ and $\alpha \neq \beta$. Define the bivariate Lévy measure $\nu$ using the Lévy copula $C(x_1, x_2) = \min(x_1, x_2)$. Then the tail integral of $\nu$ is given by $U(x_1, x_2) = \min(x_1^{-\alpha}, x_2^{-\beta})$. But this implies that the Lévy measure $\nu$ is concentrated on the curve $x_2 = x_1^{\alpha/\beta}$. In particular, its radial component cannot have a Lebesgue density, so $\nu$ is not selfdecomposable. However, the marginals $\nu_1$ and $\nu_2$ of $\nu$ are $\alpha$- and $\beta$-stable, respectively, and hence selfdecomposable.

In certain settings it is however possible to obtain desirable additional properties. Below we describe a general framework for this and we exemplify by indicating how to construct models in some important subclasses of the set of all infinitely divisible distributions on $\mathbb{R}^m$.

## 4.1 Construction of Lévy copulas with further probabilistic properties

Suppose that, for $m = 1, 2, ...$, we have a one-to-one mapping $\Psi_0^{(m)}$ that is defined on the class $\mathcal{L}^m$ of Lévy measures (or perhaps on a major subclass $\mathcal{D}^m$ of $\mathcal{L}^m$) and whose range is some subclass $\mathcal{A}^m$ of $\mathcal{L}^m$ possessing interesting probabilistic properties. In particular, then $\Psi_0^{(m)}$ sends each of the one-dimensional marginals of a Lévy measure $\nu$ into the corresponding one-dimensional marginal of $\Psi_0^{(m)}(\nu)$, that is, denoting by $\Pi_i$ the projection onto the $i$-th axis of $\mathbb{R}^m$, for each $i$ we have a mapping $\Pi_i\nu \to \Pi_i\tilde{\nu}$. We assume that these latter mappings are all effected by $\Psi_0^{(1)}$ in the sense that

$$\Pi_i \Psi_0^{(m)}(\nu) = \Psi_0^{(1)} \Pi_i(\nu). \tag{4.1}$$

We shall refer to such mappings as *mappings of Upsilon-type*, cf. Barndorff-Nielsen and Thorbjørnsen [8] Now suppose furthermore that $\mathcal{D}^1 = \Pi_i \mathcal{D}^m$, $\mathcal{A}^1 = \Pi_i \mathcal{A}^m$, $i = 1, 2, ..., m = 1, 2, ....$ Under these assumptions, we can now construct a model in $\mathcal{A}^m$ with prescribed marginals $\tilde{\nu}_i$ in $\mathcal{A}^1$, $i = 1, 2, ...,$ as follows. Using the inverse mapping $(\Psi_0^{(1)})^{-1}$ of $\Psi_0^{(1)}$, let

$$\nu_i = (\Psi_0^{(1)})^{-1}(\tilde{\nu}_i),$$

$i = 1, ..., m$, take any Lévy copula $C$, define $\nu$ as the Lévy measure in $\mathcal{L}^m$ determined by $C$ and by $\nu_i$, $i = 1, ..., m$, and, finally, let $\tilde{\nu} = \Psi_0^{(m)}(\nu)$. To illustrate the construction scheme further, denote for any Lévy copula $C$ by $T_C : (\mathcal{L}_+^1)^m \to \mathcal{L}_+^m$ the mapping which combines $m$ one-dimensional Lévy measures via (1.4) to an $m$-dimensional Lévy measure with given one-dimensional margins and Lévy copula $C$. In the present situation, the Lévy copula $\widetilde{C}$ coupling $\widetilde{\nu}_1, \ldots, \widetilde{\nu}_m \in \mathcal{A}^1$ to give a Lévy measure $\widetilde{\nu} \in \mathcal{A}^m$ with margins $\widetilde{\nu}_1, \ldots, \widetilde{\nu}_m$ is then found by the following commuting diagram; here, $C$ is any Lévy copula:

$$\begin{array}{ccc} \widetilde{\nu}_1, \ldots, \widetilde{\nu}_m & \stackrel{(\Psi_0^{(1)})^{-1}}{\longrightarrow} & \nu_1, \ldots, \nu_m \\ \Big\downarrow T_{\widetilde{C}} & & \Big\downarrow T_C \\ \widetilde{\nu} & \stackrel{\Psi_0^{(m)}}{\longleftarrow} & \nu \end{array} \tag{4.2}$$

Offhand, it is not obvious that there do exist examples of tractable one-to-one mappings $\Psi_0^{(m)}$ with associated interesting image sets $\mathcal{A}^m$, of the kind described above. However, some recent investigations, arising out of a study of free probability (see Barndorff-Nielsen and Thorbjørnsen [6, 7, 8] and Barndorff-Nielsen, Maejima and Sato [3]), have shown that this is in fact the case. In Sections 4.2 – 4.4 we shall describe three such examples, starting by describing the image sets $\mathcal{A}^m$. In all three cases the mapping $\Psi_0^{(m)}$ is of the form

$$\tilde{\nu}(B) = \int_0^\infty \nu(\xi^{-1}B)\tau(\mathrm{d}\xi) \quad \forall\, B \text{ Borel set in } \mathbb{R}^m, \tag{4.3}$$

for some Radon measure $\tau$ on $(0, \infty)$. In the actual settings, $\tau$ has a density $\psi$ with respect to Lebesgue measure.

A systematic study of measure transformations having the structure (4.3) has been initiated, see Barndorff-Nielsen and Thorbjørnsen [8, Section 3.6]. A point of particular probabilistic interest is that in many cases, including the three discussed in Sections 4.2 – 4.4 below, there exists a stochastic representation of (4.3) in terms of an integral of a deterministic function with respect to the Lévy process determined by $\nu$.

## 4.2 The Goldie-Steutel-Bondesson class

Based on the work of Goldie [12] and Steutel [18], Bondesson [9] considered the smallest class of probability distributions on $[0, \infty)$ which is closed under weak convergence and convolution and contains all mixtures of exponential distributions. This class was extended to distributions on the real line, and we shall refer to that as the Goldie-Steutel-Bondesson class $B(\mathbb{R})$, or G-S-B class, for short. Barndorff-Nielsen, Maejima and Sato [3] generalised this further to distributions on $\mathbb{R}^m$: by definition, the *multivariate G-S-B class* $B(\mathbb{R}^m)$ consists of all infinitely divisible distributions $\mu$ whose Lévy measure $\nu$ can be expressed as

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\, l_\xi(r)\, dr \quad \forall\, B \text{ Borel set in } \mathbb{R}^m \setminus \{0\}. \tag{4.4}$$

Here, $\lambda$ is a positive measure on $S = \{\xi \in \mathbb{R}^m : |\xi| = 1\}$ and $(l_\xi)_{\xi \in S}$ is a family of functions on $(0, \infty)$ such that $l_\xi(r)$ is completely monotone in $r$ for $\lambda$-a.e. $\xi$, and $l_\xi(r)$ is measurable in $\xi$ for each $r > 0$. A characterisation of $B(\mathbb{R}^m)$ as the smallest class closed under weak convergence and convolution and containing all "elementary mixtures" of signed exponential random variables in $\mathbb{R}^m$ was also obtained in [3]; we shall not make use of this characterisation in the sequel.

We shall be interested in the subclass $B(\mathbb{R}_+^m)$, consisting of all elements of $B(\mathbb{R}^m)$ whose *Lévy measure* is concentrated on $\mathbb{R}_+^m$. For notational convenience, since for any infinitely divisible distribution $\mu$ the property of belonging to $B(\mathbb{R}_+^m)$ is completely determined by its Lévy measure $\nu$, we shall also say that $\nu$ belongs to $B(\mathbb{R}_+^m)$. In one dimension, $B(\mathbb{R}_+)$ consists of all infinitely divisible distributions whose Lévy measure is concentrated on $(0, \infty)$ and has a completely monotone Lévy density there.

It follows from (4.4) that $\mathcal{A}^m := B(\mathbb{R}_+^m)$ satisfies $\mathcal{A}^1 = \Pi_i \mathcal{A}^m$ for every $m \in \mathbb{N}$. To construct multivariate Lévy measures in the G-S-B class with prescribed margins in the G-S-B class, we will apply the general method outlined in Section 4.1. The role of the transform $\Psi_0^{(m)}$ is played by the

Upsilon transformation $\Upsilon_0^{(m)}$, defined for $\nu \in \mathcal{L}^m$ by

$$\widetilde{\nu}(B) = \Upsilon_0^{(m)}(B) := \int_0^\infty e^{-s}\,\nu(s^{-1}B)\,ds, \quad B \text{ Borel set in } \mathbb{R}^m. \tag{4.5}$$

This falls into the general class of transformations considered in (4.3), with density $\psi(\xi) = e^{-\xi}1_{(0,\infty)}(\xi)$ of the Radon measure $\tau$.

The mapping $\Upsilon_0^{(m)}$ was introduced by Barndorff-Nielsen and Thorbjørnsen [6, 7] for the one-dimensional case $m = 1$ and extended by Barndorff-Nielsen, Maejima and Sato [3] to the multivariate setting. There it was shown that $\Upsilon_0^{(m)}$ is a bijection from the class of $m$-dimensional Lévy measures onto the Lévy measures in the G-S-B class $B(\mathbb{R}^m)$. Since every Lévy measure in the G-S-B class has a radial Lévy density, $\Upsilon_0^{(m)}$ can be seen as a regularizer, in particular it follows that the Lévy measure constructed in Example 4.1 is not in the G-S-B class $B(\mathbb{R}_+^2)$, although its margins are. This means that not every Lévy copula coupled with margins in the 1-dimensional G-S-B class gives Lévy measures in the $m$-dimensional G-S-B class, but the scheme outlined in Section 4.1 works, since it follows easily from (4.5) that $\Pi_i\Upsilon_0^{(m)} = \Upsilon_0^{(1)}\Pi_i$ for $i = 1, \dots, m$, so that (4.1) holds for $\Psi_0^{(m)} = \Upsilon_0^{(m)}$.

To illustrate the construction method in more detail, suppose that $\widetilde{\nu}_1, \dots, \widetilde{\nu}_m \in B(\mathbb{R}_+)$ are prescribed marginal Lévy measures. Denote their Lévy densities by $\widetilde{f}_i$, $i = 1, \dots, m$. Since the $\widetilde{f}_i$ are completely monotone, by Bernstein's theorem there exist positive measures $\phi_i$ on $(0, \infty)$ such that $\widetilde{f}_i$ is the Laplace transform of $\phi_i$, i.e.

$$\widetilde{f}_i(x_i) = \int_{(0,\infty)} e^{-x_i s}\,d\phi_i(s), \quad x_i > 0.$$

Suppose now that the measures $\phi_i$ have a density, and denote this density by $\phi_i(ds) = s\,h_i(s)\,ds$ for some function $h_i$. Then, as shown in Barndorff-Nielsen and Thorbjørnsen [6], it follows that the tail integral $U_i$ of $(\Upsilon_0^{(1)})^{-1}$ is given by

$$U_i(x_i) = \int_0^{1/x_i} h_i(s)\,ds, \quad x_i > 0. \tag{4.6}$$

Then if $C$ is any Lévy copula, define the tail integral $U$ of a Lévy measure $\nu$ by (1.4), and let $\widetilde{\nu} := \Upsilon_0^{(m)}(\nu)$. The measure $\widetilde{\nu}$ is then in the G-S-B class

$B(\mathbb{R}^m_+)$ with margins $\widetilde{\nu}_1, \dots, \widetilde{\nu}_m$, and its tail integral $\widetilde{U}$ is given by

$$\begin{aligned}\widetilde{U}(x_1, \dots, x_m) &= \int_0^\infty e^{-s} U(x_1/s, \dots, x_m/s)\, ds \\ &= \int_0^\infty e^{-s} C(U_1(x_1/s), \dots, U_m(x_m/s)), \quad x_1, \dots, x_m \geq 0. \end{aligned} \tag{4.7}$$

**Example 4.2** Suppose that $\widetilde{\nu}_i$, $i = 1, \dots, m$, are Lévy measures of inverse Gaussian distributions $IG(\delta_i, \gamma_i)$ with $\delta_i, \gamma_i > 0$. Their Lévy density $\widetilde{f}_i$ is given by

$$\widetilde{f}_i(x_i) = \frac{\delta_i}{\sqrt{2\pi}}\, x_i^{-3/2} \exp\{-\frac{1}{2}\gamma_i^2 x_i\}\, 1_{(0,\infty)}(x_i), \quad x_i > 0,$$

see Barndorff-Nielsen and Shephard [4]. Define $h_i(s)$ by

$$h_i(s) := \frac{\delta_i \sqrt{2}}{\pi} \frac{s - \gamma_i^2/2}{s} 1_{[\gamma_i^2/2,\infty)}(s), \quad s > 0.$$

Then $\widetilde{f}_i$ is the Laplace transform of $s \mapsto s\, h_i(s)$, see e.g. [1], formula 29.3.63. The tail integrals $U_i$ of $(\Upsilon_0^{(1)})^{-1}$ can then by calculated using (4.6), giving

$$U_i(x_i) = \begin{cases} 0, & x_i \geq 2/\gamma_i^2, \\ \frac{\delta_i\sqrt{2}}{\pi} \left\{ 2(x_i^{-1} - \gamma_i^2/2)^{1/2} - \sqrt{2}\gamma_i \arctan \frac{\sqrt{2}(x_i^{-1}-\gamma_i^2/2)^{1/2}}{\gamma_i} \right\}, & x_i \leq 2/\gamma_i^2. \end{cases}$$

Then if $C$ is any Lévy copula, (4.7) defines the tail integral of a Lévy measure in the G-S-B class with $IG(\delta_i, \gamma_i)$ margins.

## 4.3 The Lévy class

The Lévy class $L(\mathbb{R}^m)$ is the class of all selfdecomposable distributions on $\mathbb{R}^m$. Recall that an infinitely divisible distribution $\mu$ is self-decomposable if and only if its Lévy measure $\nu$ has representation (4.4), where $l_\xi$ does not need to be completely monotone, but $r \mapsto r l_\xi(r)$ has to be decreasing on $(0, \infty)$, see Sato [16], Theorem 15.10. By an abuse of language, we shall also say that the Lévy measure $\nu$ is self-decomposable. Denote by $L(\mathbb{R}^m_+)$ the class of self-decomposable distributions with Lévy measure in $\mathcal{L}^m_+$, by $ID_{\log}(\mathbb{R}^m)$ the class of infinitely divisible distributions $\mu$ which satisfy $\int_{|x|>1} \log|x|\, d\mu(x) < \infty$, and by $ID_{\log}(\mathbb{R}^m_+)$ the distributions in $ID_{\log}(\mathbb{R}^m)$ whose Lévy measure is in

$\mathcal{L}_+^m$. If $\mu$ is infinitely divisible with Lévy measure $\nu \in \mathcal{L}^m$, then $\mu \in ID_{\log}(\mathbb{R}^m)$ if and only if $\int_{|x|>1} \log|x|\, d\nu(x) < \infty$, see Sato [16], Theorem 25.3; we shall also write $\nu \in ID_{\log}(\mathbb{R}^m)$, $\nu \in L(\mathbb{R}^m)$ and similar expressions if additionally the Lévy measure is in $\mathcal{L}_+^m$.

For a Lévy measure $\nu \in ID_{\log}(\mathbb{R}^m)$, define the mapping $\Phi_0^{(m)}$ by

$$\Phi_0^{(m)}(\nu)(B) := \int_0^\infty \nu(e^s B)\, ds = \int_{-1}^1 s^{-1}\, \nu(s^{-1}B)\, ds \quad \forall\, B \text{ Borel set in } \mathbb{R}^m.$$

The mapping $\Phi_0^{(m)}$ has the general form (4.3), where the measure $\tau$ has density $\psi(\xi) = \xi^{-1}\, 1_{[-1,1]}(\xi)$. Sato and Yamazato [17], Section 4, showed that $\Phi_0$ defines a bijection from the Lévy measures in $ID_{\log}(\mathbb{R}^m)$ to those in $L(\mathbb{R}^m)$, hence also from $ID_{\log}(\mathbb{R}_+^m)$ onto $L(\mathbb{R}_+^m)$. Denote $\mathcal{A}^m := L(\mathbb{R}_+^m)$. Then $\Pi_i \mathcal{A}^m = \mathcal{A}^1$. Furthermore, easy calculations show that if $\nu$ is a Lévy measure with marginals $\nu_1, \ldots, \nu_m$, then $\nu \in ID_{\log}(\mathbb{R}_+^m)$ if and only if $\nu_i \in ID_{\log}(\mathbb{R}_+)$ for all $i = 1, \ldots, m$. In particular, any Lévy copula applied to margins in $ID_{\log}(\mathbb{R}_+)$ gives a Lévy measure in $ID_{\log}(\mathbb{R}_+^m)$. Since the mapping $\Phi_0$ commutes with projection, i.e. (4.1) holds for $\Psi_0 = \Phi_0$, the method outlined in Section 4.1 can be applied to construct Lévy measures in $L(\mathbb{R}_+^m)$ with prescribed selfdecomposable margins. In more detail, let $\widetilde{\nu}_1, \ldots, \widetilde{\nu}_m$ be Lévy measures in $L(\mathbb{R}_+^m)$. Denote their Lévy density by $\widetilde{f}_i$, $i = 1, \ldots, m$. Then the tail integral of $\nu_i := (\Phi_0^{(1)})^{-1}(\widetilde{\nu}_i)$ (i.e. of the background driving Lévy process) is given by

$$U_i(x_i) = x_i\, \widetilde{f}_i(x_i), \quad x_i > 0,$$

see Barndorff-Nielsen and Shephard [5]. Thus if $C$ is any Lévy copula, then (1.4) defines the tail integral of a Lévy measure $\nu \in ID_{\log}(\mathbb{R}_+^m)$, and $\widetilde{\nu} := \Phi_0^{(m)}(\nu)$ is in $L(\mathbb{R}_+^m)$, with margins $\widetilde{\nu}_1, \ldots, \widetilde{\nu}_m$. Its tail integral $\widetilde{U}$ is given by

$$\begin{aligned} \widetilde{U}(x_1, \ldots, x_m) &= \int_0^\infty U(e^s x_1, \ldots, e^s x_m)\, ds \\ &= \int_0^\infty C(e^s x_1\, \widetilde{f}_1(e^s x_1), \ldots, e^s x_m\, \widetilde{f}_m(e^s x_m))\, ds, \quad x_1, \ldots, x_m > 0. \end{aligned}$$

## 4.4 The Thorin class

In Barndorff-Nielsen, Maejima and Sato [3], the $m$-dimensional Thorin class $T(\mathbb{R}^m)$ is defined to be the class of all infinitely divisible distributions $\mu$

whose Lévy measure $\nu$ has representation (4.4), where $r \mapsto r l_\xi(r)$ has to be completely monotone on $(0, \infty)$. We shall also write $\nu \in T(\mathbb{R}^m)$. The $m$-dimensional Thorin class $T(\mathbb{R}^m)$ is a generalisation of the one-dimensional Thorin class $T(\mathbb{R})$ introduced by Thorin [20]. It can be shown that $T(\mathbb{R}^m)$ is a proper subclass of $B(\mathbb{R}^m) \cap L(\mathbb{R}^m)$. A probabilistic interpretation as for the G-S-B class is given in [3]. There, it is also shown that the Lévy measures in $T(\mathbb{R}^m)$ constitute the image set of Lévy measures in $L(\mathbb{R}^m)$ under $\Upsilon_0^{(m)}$ (for $m = 1$ this was proved in [6]), and also the image set of Lévy measures in $B(\mathbb{R}^m) \cap ID_{\log}(\mathbb{R}^m)$ under $\Phi_0^{(m)}$. Furthermore, $\Phi_0^{(m)}$ and $\Upsilon_0^{(m)}$ commute, i.e. $\Phi_0^{(m)} \Upsilon_0^{(m)}(\mu) = \Upsilon_0^{(m)} \Phi_0^{(m)}(\mu)$ for $\nu \in ID_{\log}(\mathbb{R}^m)$, and $\Phi_0^{(m)} \Upsilon_0^{(m)}$ has the general form (4.3) where the measure $\tau$ has density $\psi(\xi) = \xi^{-1} e^{-\xi} 1_{(0,\infty)}(\xi)$. Denote by $T(\mathbb{R}_+^m)$ the class of all infinitely divisible distributions in the Thorin class whose Lévy measure $\nu$ is in $\mathcal{L}_+^m$, and write also $\nu \in T(\mathbb{R}_+^m)$. Then the general scheme outlined in Section 4.1 can be applied to construct Lévy measures $\widetilde{\nu}$ in $T(\mathbb{R}_+^m)$ with prescribed margins $\widetilde{\nu}_1 \in T(\mathbb{R}_+)$. Alternatively, the results of Sections 4.2 and 4.3 can be used: take the inverses $\nu_i$ of the marginal Lévy measures $\widetilde{\nu}_i$ under $\Upsilon_0^{(1)}$, construct a Lévy measure $\nu \in L(\mathbb{R}_+^m)$ with margins $\nu_i$ as in Section 4.3, and set $\widetilde{\nu} := \Upsilon_0^{(m)}(\nu)$. Similarly, one can apply first $(\Phi_0^{(1)})^{-1}$, then use the results of Section 4.2, and finally apply $\Phi^{(m)}$ to the Lévy measure obtained in this way.

## 4.5 Transformation of Lévy copulas under $\Upsilon_0$

In Section 4.1 we have seen how Lévy copulas and $\Upsilon$-type transformations $\Psi_0$ can be used to construct multivariate Lévy measures $\widetilde{\nu}$ with given margins and further properties. It is interesting to compare properties of the Lévy copula $C$ associated with $\nu = (\Psi_0^{(m)})^{-1}$ with the Lévy copula $\widetilde{C}$ associated with $\widetilde{\nu}$, where the relation between $C$ and $\widetilde{C}$ is given by the diagram (4.2). We shall restrict ourselves to the mapping $\Upsilon_0$ appearing in Section 4.2. A natural question is e.g. whether a homogeneous Lévy copula $C$ gives rise to a homogeneous Lévy copula $\widetilde{C}$, or if the Lévy copulas of complete dependence and independence are preserved, respectively. For the Lévy copula of independence this is indeed the case, as follows easily from (4.7). On the other hand, if $\nu \in \mathcal{L}_+^2$ with the (homogeneous) Lévy copula $C(u_1, u_2) = \min\{u_1, u_2\}$ of complete dependence, and with tail integrals $U_1(x_1) = 2x_1^{-1}$ for $x_1 \geq 1/2$, $U_1(x_1) = 3 + (2x_1)^{-1}$ for $x_1 < 1/2$, and $U_2(x_2) = x_2^{-1}$ for $x_2 > 0$, then easy calculations show that the Lévy copula $\widetilde{C}$ associated with $\widetilde{\nu} = \Upsilon_0^{(2)}(\nu)$ is not homogeneous. In particular, the Lévy copula of complete dependence is not preserved, nor is homogeneity of Lévy copulas. The Lévy copula $\widetilde{C}$ depends not only on $C$, but also on the margins $\nu$. However, if the margins of $\nu$ are stable, then we have the following positive result:

**Theorem 4.3** *Let $\nu \in \mathcal{L}_+^m$ have stable non-degenerate margins with indices $\kappa_1, \ldots, \kappa_m \in (0, 2)$. Then the Lévy copula associated with $\nu$ is homogeneous if and only if the Lévy copula $\widetilde{C}$ associated with $\widetilde{\nu} = \Upsilon_0(\nu)$ is homogeneous.*

**Proof.** Let $\nu_i$, $i = 1, \ldots, m$, be the non-degenerate $\kappa_i$ stable margins of $\nu$. Then it is easy to see that

$$\widetilde{C}(u_1, \ldots, u_m) = \int_0^\infty e^{-s}\, C\left(s^{\kappa_1} \frac{u_1}{\Gamma(\kappa_1 + 1)}, \ldots, s^{\kappa_m} \frac{u_m}{\Gamma(\kappa_m + 1)}\right) ds. \quad (4.8)$$

From this follows immediately that if $C$ is homogeneous then so is $\widetilde{C}$. For the converse, suppose that $\widetilde{C}$ is homogeneous. Equation (4.8) then shows, that for any $t > 0$,

$$t^{-1}\widetilde{C}(t^{-\kappa_1}u_1, \ldots, t^{-\kappa_m}u_m) = \int_0^\infty e^{-rt}\, C\left(\frac{r^{\kappa_1}u_1}{\Gamma(\kappa_1 + 1)}, \ldots, \frac{r^{\kappa_m}u_m}{\Gamma(\kappa_m + 1)}\right) dr.$$

For fixed $u = (u_1, \ldots, u_m) \in [0, \infty]^m \setminus \{(\infty, \ldots, \infty)\}$, define

$$\begin{aligned} f_u &: (0, \infty) \to \mathbb{R}, \qquad r \mapsto C\left(r^{\kappa_1} \frac{u_1}{\Gamma(\kappa_1 + 1)}, \ldots, r^{\kappa_m} \frac{u_m}{\Gamma(\kappa_m + 1)}\right), \\ g_u &: (0, \infty) \to \mathbb{R}, \qquad t \mapsto t^{-1}\widetilde{C}\left(t^{-\kappa_1}u_1, \ldots, t^{-\kappa_m}u_m\right). \end{aligned}$$

Then $g_u$ is the Laplace transform of $f_u$, $g_u = Lap(f_u)$. Further, for fixed $s > 0$, $\frac{1}{s}g_{su} = Lap(\frac{1}{s}f_{su})$. Now if $\widetilde{C}$ is homogeneous, then $g_u = \frac{1}{s}g_{su}$. From the uniqueness theorem for Laplace transforms then follows that $\frac{1}{s}f_{su}(r) = f_u(r)$ almost everywhere in $r$, and even everywhere in $r$ since both functions are continuous by (2.5). In particular, $\frac{1}{s}f_{su}(1) = f_u(1)$, showing that $C$ is homogeneous. ■

One might wonder if both $\nu$ and $\Upsilon_0^{(m)}(\nu)$ having homogeneous Lévy copulas implies stability of the margins. This, however, is not the case:

**Example 4.4** Let $\nu \in \mathcal{L}_+^m$ with marginal tail integrals $U_1(x) \geq U_2(x) \geq \ldots \geq U_m(x)\ \forall\ x \in [0,\infty]$ and associated Lévy copula $C(u_1,\ldots,u_m) = \min\{u_1,\ldots,u_m\}$. Then it follows easily that $\widetilde{C} = C$ is associated with $\Upsilon_0^{(m)}(\nu)$. In particular, $C$ and $\widetilde{C}$ are both homogeneous, although the margins of $\nu$ are not necessarily stable.

# 5 The Lévy measure induced by Lévy copulas

An interesting feature of distributional copulas is that they themselves are distribution functions with special properties. It is natural to ask whether Lévy copulas can be interpreted as a special class of Lévy copulas, too. This is indeed the case when a simple "inversion map" is applied to the measure $\chi_C$ defined by (1.2):

**Definition 5.5** *For $m \in \mathbb{N}$, denote by $Q = Q_m$ the bijection*

$$Q_m : [0,\infty]^m \to [0,\infty]^m, \quad (x_1,\ldots,x_m) \mapsto (x_1^{-1},\ldots,x_m^{-1}),$$

*where $1/0$ has to be interpreted as $\infty$, and $1/\infty$ as $0$. For any Lévy copula $C$ on $[0,\infty]^m$, the measure $\nu_C$ is then defined as the image of the measure $\chi_C$ given in (1.2), under the mapping $Q_m$, i.e.*

$$\nu_C(B) = (Q\chi_C)(B) = \chi_C(Q_m^{-1}(B)) \quad \forall\ B \text{ Borel set in } [0,\infty]^m.$$

From (1.2) we see that $\nu_C$ is thus determined by

$$\nu_C([x_1,\infty]\times\ldots\times[x_m,\infty]) = C(x_1^{-1},\ldots,x_m^{-1}), \quad 0 \leq x_1,\ldots,x_m \leq \infty, \tag{5.9}$$

and that $\nu_C$ has no atom at $(0,\ldots,0)$. Then $\nu_C$ is a Lévy measure, more precisely we have:

**Theorem 5.6** *If $C$ is an $m$-dimensional Lévy copula, then the measure $\nu_C$ is a Lévy measure with marginal tail integrals $[0,\infty] \to [0,\infty]$, $x_k \mapsto x_k^{-1}$. In particular, then it has 1-stable margins. The Lévy measure $\nu_C$ is not of finite variation, i.e. $\int_{|x|<1} |x|\nu_C(dx) = \infty$. Conversely, if $\nu \in \mathcal{L}_+^m$ is any Lévy measure with marginal tail integrals $[0,\infty] \to [0,\infty]$, $x_k \mapsto x_k^{-1}$, then there exists a unique Lévy copula $C$ such that $\nu_C = \nu$.*

**Proof.** Let $C$ be a Lévy copula. Then groundedness of $C$ corresponds to the fact that $\nu_C$ is concentrated on $[0,\infty)^m$, and that it has no atom at the origin is due to the fact that $\chi_C$ has none at $(\infty,\ldots,\infty)$. Finiteness of $C$ on $[0,\infty]^m \setminus \{(\infty,\ldots,\infty)\}$ shows that $\nu_C$ is finite outside neighbourhoods of the origin, and it is a Lévy measure since

$$\int_{[0,1]^m} \sum_{k=1}^m x_k^2 \, d\nu_C(x_1,\ldots,x_m) \le \sum_{k=1}^m \int_{[0,1]} x_k^2 \, d(\nu_C)_k(x_k) = \sum_{k=1}^\infty \int_1^\infty \frac{1}{y_k^2} \, dy_k < \infty$$

(here, $(\chi_C)_k$ denotes the $k$-th marginal measure of $\chi_C$). That $\nu_C$ is not of finite variation can be seen from

$$\int_{[0,1]^m} \sum_{k=1}^m x_k \, d\nu_C(x_1,\ldots,x_m) \ge \int_{[1,\infty]^m} \frac{1}{y_1} \, d\chi_C(y_1,\ldots,y_m) =$$

$$\int_{[1,\infty]\times[0,\infty]^{m-1}} \frac{1}{y_1} d\chi_C(y_1,\ldots,y_m) - \int_{[1,\infty]\times([0,\infty]^{m-1}\setminus[1,\infty]^{m-1})} \frac{1}{y_i} d\chi_C(y_1,\ldots,y_m);$$

here, the first integral is equal to $\int_1^\infty \frac{1}{y_1} \, dy_1 = \infty$, while the second integral is finite since $\chi_C([1,\infty] \times ([0,\infty]^{m-1} \setminus [1,\infty]^{m-1})) < \infty$. The remaining assertions are clear. ■

**Remark 5.7** *Let $C$ be a Lévy copula, and denote the (marginal) tail integrals of $\nu_C$ by $U_C$ and $U_{C,i}$, respectively. Then it follows from* (5.9) *and* (1.4) *that for $x_1,\ldots,x_m \in [0,\infty]$,*

$$U_C(x_1,\ldots,x_m) = C(x_1^{-1},\ldots,x_m^{-1}) = C(U_{C,1}(x_1),\ldots,U_{C,m}(x_m)).$$

*This shows that the Lévy copula, associated with the Lévy measure $\nu_C$ by* (1.4)*, is again $C$.*

The mapping $Q_m$ is not the only bijection that could have been used to transform $\chi_C$ (and hence Lévy copulas) to Lévy measures. However, the mapping $Q_m$ has many appealing features. Apart from being easy to calculate, the obtained Lévy measure reflects the uniform margins property of distributional Lévy copulas in a very clear way having tail integrals $[0,\infty] \to [0,\infty]$, $x_k \mapsto x_k^{-1}$. Further, as we shall see in Theorem 5.9, homogeneous Lévy copulas $C$ are precisely those Lévy measures for which $\nu_C$ is the Lévy measure of a 1-stable distribution, and hence this allows an easy description of homogeneous Lévy copulas as done in Theorem 3.4. Finally, the mapping $Q_m$ appears naturally when dealing with the Upsilon transform $\Upsilon_0^m$, as can be seen from (4.7).

The following proposition provides a stepping stone to Theorem 5.9 below.

**Proposition 5.8** *Let $\alpha \in (0,2)$ and $\nu$ be a Lévy measure with non-degenerate $\alpha$-stable margins and associated Lévy copula $C$. Then*
*(a) $\nu$ is stable if and only if $\nu_C$ is 1-stable.*
*(b) $\nu$ is selfdecomposable if and only if $\nu_C$ is selfdecomposable.*

**Proof.** We first prove part $(b)$. Let $U_i(x_i) = k_i^{-1} x_i^{-\alpha}$ ($k_i > 0$, $i = 1, \ldots, m$) be the marginal tail integrals of $\nu$. By Sato [16], Theorem 15.8, $\nu$ is selfdecomposable if and only if $\nu(t^{-1}B) \geq \nu(B)$ for all Borel sets $B$ in $[0,\infty)^m$ and all $t \geq 1$, or what is the same if

$$\chi(t^{-1}B) \leq \chi(B) \tag{5.10}$$

for all Borel sets $B$ in $(0,\infty]^m$ and all $t \geq 1$; here $\chi = Q_m \nu$. It is enough to check (5.10) for all Borel sets of the form $B := (a_1, b_1] \times \ldots \times (a_m, b_m]$. With the aid of the tail integral of $\nu$ we can write

$$\begin{aligned} \chi(B) &= \sum \operatorname{sgn}(c)\, U(Q_m(c)) \\ &= \sum \operatorname{sgn}(c)\, C(k_1\, x_1^\alpha, \ldots, k_m\, x_m^\alpha), \end{aligned}$$

where the sum is taken over all vertices $c = (c_1, \ldots, c_m)$ of $B$. Thus, $\nu$ is selfdecomposable if and only if

$$\sum \operatorname{sgn}(c)\, C(t^{-\alpha} k_1 c_1^\alpha, \ldots, t^{-\alpha} k_m c_m^\alpha) \leq \sum \operatorname{sgn}(c)\, C(k_1 c_1^\alpha, \ldots, k_m c_m^\alpha)$$

for all $t \geq 1$ and all sets $(a_1, b_1] \times \ldots \times (a_m, b_m]$. Substituting $u_i = k_i a_i^\alpha$, $v_i = k_i b_i^\alpha$, this is the same as

$$\sum \operatorname{sgn}(d)\, C(t^{-\alpha} d) \leq \sum \operatorname{sgn}(d)\, C(d),$$

where the sum ranges over all vertices $d$ of $(u_1, v_1], \ldots, (u_m, v_m]$. The latter is the condition for $\chi_C$ to satisfy (5.10), i.e. for $\nu_C$ to be selfdecomposable.

The proof of $(a)$ is similar, using Sato [16], Theorem 14.3. ■

Tankov [19] showed that if $\alpha \in (0,2)$ and if a positive Lévy measure $\nu$ has non-degenerate $\alpha$-stable margins, then $\nu$ is $\alpha$-stable if and only if the associated Lévy copula is homogeneous. Combining this with the previous proposition and Remark 5.7, we immediately obtain:

**Theorem 5.9** *A Lévy copula $C$ is homogeneous if and only if $\nu_C$ is a 1-stable Lévy measure.*

Using this, Theorem 3.4 follows easily:

**Proof of Theorem 3.4.** It is easy to see that every function $C$ satisfying (3.2) and (3.3) is a Lévy copula. For the converse, the spectral representation of stable Lévy measures (cf. Sato [16], Theorem 14.3) shows that the Lévy measure $\nu_C$ coming from a Lévy copula $C$ is 1-stable if and only if it has the representation

$$\nu_C(B) = \int_{S_+} \int_0^\infty 1_B(r\xi)\, r^{-2}\, dr\, \lambda(d\xi) \quad \forall\, B \text{ Borel set in } [0,\infty)^m,$$

where $\lambda$ is a finite measure on $S_+$. As can be easily seen, this is equivalent to (3.3). Equation (3.2) corresponds to the uniform margins property of Lévy copulas. ■

Returning to the characterisation of convergence of Lévy measures in Theorem 2.1, it is natural to ask whether the pointwise convergence condition of $C^{(n)}$ there can be replaced by vague convergence of $\nu_{C^{(n)}}$. Since the limit Lévy copula $C$ in Theorem 2.1 is not necessarily unique if $\operatorname{Ran} U_i \neq [0,\infty]$ for some $i$, vague convergence is not to be expected in general. However, if $\operatorname{Ran} U_i = [0,\infty]$ for all $i = 1,\ldots,m$, then the statement on the pointwise convergence of $C^{(n)}$ in Theorem 2.1 can be replaced by vague convergence of $\nu_{C^{(n)}}$. This follows from the following lemma, which is an immediate consequence of the characterisation of vague convergence in terms of pointwise convergence of tail integrals, as stated before Theorem 2.1.

**Lemma 5.10** *Let $(C^{(n)})_{n\in\mathbb{N}}$ and $C$ be Lévy copulas. Then $C^{(n)}$ converges pointwise on $[0,\infty]^m$ to $C$ if and only if $\nu_{C^{(n)}} \xrightarrow{\#} \nu_C$ as $n \to \infty$.*

Finally, let us return to the transformation of Lévy copulas under the mapping $\Upsilon_0^m$, as discussed in Section 4.5. We have seen that the Lévy copula of a transformed Lévy measure does not need to share the same properties as the original Lévy copula. However, if the margins of the Lévy measure are stable, then for example homogeneity of Lévy copulas is preserved under $\Upsilon_0^m$. Since every Lévy copula $C$ defines a Lévy measure $\nu_C$ with 1-stable margins, to which $C$ is again associated via (1.4), it is natural to define a direct Upsilon transformation acting on Lévy copulas:

**Definition 5.11** For any Lévy copula $C$, the transformed Lévy copula $\Upsilon_0^{cop}(C)$ is defined by

$$\Upsilon_0^{cop}(C)(u_1,\dots,u_m) = \int_0^\infty e^{-s}\, C(su_1,\dots,su_m)\, ds \quad \forall\, u_1,\dots,u_m \in [0,\infty].$$

Then $\Upsilon_0^{cop}(C)$ is the Lévy copula of $\Upsilon_0^{(m)}(\nu_C)$, as can be seen from (4.8). Note that $\Upsilon_0^{cop}(C)$ can be defined for any Lévy copula $C$, while $\widetilde{C}$ as appearing in Section 4.5 depends on the margins of a Lévy measure.

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