Lévy-driven time series models for financial data

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Abstract

The ARCH and GARCH models of Engle (1982) and Bollerslev (1986) respectively have had great success in the modelling of financial time series. Discrete-time stochastic volatility models have also been found to be very useful in representing the time-variation of volatility observed in such data. In this review we discuss Lévy-driven continuous-time versions of these processes and some related inference questions.

Keywords: Lévy process, Lévy-driven CARMA process, stochastic volatility, COGARCH process, generalized Ornstein-Uhlenbeck process

AMS Classification : 62M10, 60H10, 91G70

1 Introduction

The study of time series with continuous time parameter received great impetus from the very successful application of such processes in theoretical finance, particularly with the work of Black, Scholes and Merton on the pricing of options and the subsequent explosive growth of financial mathematics. An excellent recent overview of financial time series can be found in the book edited by Andersen et al. [2]. For further applications of Lévy processes in finance we recommend also the books of Cont and Tankov [20] and Schoutens [37].

In this article we focus attention on some financial time series driven by Lévy processes, the essential properties of which are introduced in Section 2. Lévy processes play a central role for several reasons, one being that their sample-paths are not restricted to be continuous and another that the distributions of their increments can be any of the very large

class of infinitely divisible distributions. In Section 3 we discuss Lévy-driven continuous-time autoregressive moving average (CARMA) processes which play a role in continuous time analogous to their discrete-time counterparts, and introduce the stationary Lévy-driven Ornstein-Uhlenbeck process as a special case. Section 4 deals with the celebrated continuous-time stochastic volatility model of Barndorff-Nielsen and Shephard [5] in which the volatility is a stationary Ornstein-Uhlenbeck process driven by a *subordinator* (a Lévy process with non-decreasing sample-paths). In Section 5 we consider an extended version of the Barndorff-Nielsen-Shephard model in which the volatility is a non-negative CARMA process and discuss parameter estimation for the volatility based on realized integrated volatility. In Section 6 we introduce the generalized Ornstein-Uhlenbeck (GOU) process, which generalizes the Ornstein-Uhlenbeck process in a direction different from that of the CARMA process. Finally in Section 7 we discuss the COGARCH(1,1) process of Klüppelberg et al. [29], in which the volatility is a GOU process, and the higher-order COGARCH(p,q) process of Brockwell et al. [14].

2 Lévy Processes

A *Lévy process* with values in $\mathbb{R}^d$ ($d \in \mathbb{N}$) defined on a probability space $(\Omega, \mathcal{F}, P)$ is a stochastic process $M = (M_t)_{t\geq 0}$, $M_t : \Omega \to \mathbb{R}^d$ with independent and stationary increments such that $M_0 = 0$ almost surely and the sample paths are almost surely right-continuous with finite left limits. By *independent increments*, we mean that for every $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables M_{t_0}, $M_{t_1} - M_{t_0}$, $M_{t_2} - M_{t_1}, \ldots, M_{t_n} - M_{t_{n-1}}$ are independent, and by *stationary increments* we mean that $M_{s+t} - M_s$ has the same distribution as M_t for every $s, t \geq 0$. We refer to the books by Applebaum [4], Bertoin [6], Kyprianou [30] and Sato [38] for further information about Lévy processes, in which the proofs for the results stated in this section can also be found.

Elementary examples of Lévy processes $M = (M_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ include linear deterministic processes of the form $M_t = bt$, where $b \in \mathbb{R}^d$, d-dimensional Brownian motion and d-dimensional compound Poisson processes. If $M = (M_t)_{t\geq 0}$ is any Lévy process then for all t the distribution of M_t is characterized by a unique triplet (A_M, ν_M, γ_M) consisting of a symmetric non-negative $d \times d$-matrix A_M, a measure ν_M on $\mathbb{R}^d$ satisfying $\nu_M(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min\{|x|^2, 1\}\, \nu_M(dx) < \infty$ and a constant $\gamma_M \in \mathbb{R}^d$. This triplet determines the characteristic function of M_t via the Lévy-Khintchine formula,

$$Ee^{i\langle M_t, z\rangle} = \exp\left\{ i\langle \gamma_M, z\rangle - \frac{1}{2}\langle z, A_M z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_{\{|x|\leq 1\}})\, \nu_M(dx) \right\} \quad (2.1)$$

for $z \in \mathbb{R}^d$. The measure ν_M is called the *Lévy measure* of M and A_M the *Gaussian variance*. Conversely, if $\gamma_M \in \mathbb{R}^d$, A_M is a symmetric non-negative definite $d \times d$-matrix and ν_M is a Lévy measure, then there exists a Lévy process M, unique up to identity in law, such that (2.1) holds. The triplet (A_M, ν_M, γ_M) is called the *characteristic triplet* of the Lévy process M.

For Brownian motion $(X_t)_{t\geq 0}$ with $EX_t = \mu t$ and $\mathrm{Var}(X_t) = \sigma^2 t$, the characteristic triplet is $(\sigma^2, 0, \mu)$ and for a compound Poisson process with jump-rate λ and jump-size distribution function F, the characteristic triplet is $(0, \lambda dF(\cdot), \int_{[-1,1]} \lambda x dF(x))$.

A Lévy process M with values in $\mathbb{R}^1$ is called a *subordinator* if it has increasing sample paths. This happens if and only if $A_M = 0$, $\nu_M((-\infty, 0)) = 0$ and $\int_0^1 x\, \nu_M(dx) < \infty$. Examples of subordinators include compound Poisson processes with jump distribution concentrated on $(0, \infty)$, the Gamma process and the inverse Gaussian process. The *Gamma process* with parameters $c, \lambda > 0$ is the Lévy process with characteristic triplet $(0, \nu_M, \int_0^1 ce^{-\lambda x} dx)$ and Lévy measure ν_M given by $\nu_M(dx) = cx^{-1}e^{-\lambda x}\mathbf{1}_{(0,\infty)}(x)\, dx$. For the Gamma process the distribution of M_t has Lebesgue density $x \mapsto (\Gamma(ct))^{-1}\lambda^{ct}x^{ct-1}$ $e^{-\lambda x}\mathbf{1}_{(0,\infty)}(x)$. The *inverse Gaussian* process with parameters $a, b > 0$ is defined to have characteristic triplet $A_M = 0$, Lévy measure $\nu_M(dx) = (2\pi x^3)^{-1/2} a e^{-xb^2/2}\mathbf{1}_{(0,\infty)}(x)\, dx$ and $\gamma_M = 2ab^{-1}\int_0^b (2\pi)^{-1/2} e^{-y^2/2}\, dy$. For the inverse Gaussian process the distribution of M_t has Lebesgue density $x \mapsto (2\pi x^3)^{-1/2} a t e^{-\frac{1}{2}(a^2t^2x^{-1} - 2abt + b^2x)}$.

The *jump* of a Lévy process M at time t is defined as

$$\Delta X_t := X_t - X_{t-},$$

where X_{t-} denotes the left-limit at $t > 0$ with the convention that $X_{0-} := 0$. Apart from Brownian motion with drift, every Lévy process has jumps. The Lévy measure $\nu_M(B)$ of a Borel set B describes the expected number of jumps of $(M_t)_{t\in[0,1]}$ with size in B, i.e.

$$\nu_M(B) = E \sum_{0<s\leq 1} \mathbf{1}_B(\Delta M_s).$$

A Lévy process has only finitely many jumps in finite intervals if and only if the Lévy measure of the Lévy process is finite. Every one-dimensional Lévy process is a semimartingale (cf. Applebaum [4] or Protter [36]), and its quadratic variation is given by $[M, M]_t = A_M t + \sum_{0<s\leq t} \Delta M_s^2$. We refer to Applebaum [4] and Protter [36] for further information regarding integration with respect to semimartingales or Lévy processes.

Finally, we mention that for $\kappa > 0$, a Lévy process $M = (M_t)_{t\geq 0}$ satisfies $E|M_1|^\kappa < \infty$

if and only if $E|M_t|^\kappa < \infty$ for all $t \geq 0$, which is further equivalent to $\int_{|x|\geq 1} |x|^\kappa \, \nu_M(dx) < \infty$. In particular, for $\kappa = 2$ and $d = 1$, it holds $\mathrm{Var}(M_t) = tA_M + \int_{\mathbb{R}} x^2 \, \nu_M(dx)$.

3 Lévy-driven CARMA(p, q) Processes

If $(L_t)_{t\geq 0}$ is a Lévy process with values in $\mathbb{R}$, defined as in Section 2, it can be extended to a process with stationary independent increments, right-continuous sample paths with finite left limits, $L_0 = 0$ and index set $\mathbb{R}$, by defining $L_t = -M_{-t-}, t < 0$, where $(M_t)_{t\geq 0}$ is an independent version of $(L_t)_{t\geq 0}$. Assuming this extension has been made, we define an L-driven CARMA(p, q) process with real coefficients $\{a_1, \ldots, a_p; b_1, \ldots, b_q\}$ and $p > q$ (see Brockwell [13]) as a strictly stationary solution of the suitably interpreted formal stochastic differential equation

$$a(D)V_t = b(D)DL_t, \ t \in \mathbb{R}, \tag{3.1}$$

where D denotes differentiation with respect to t,

$$\begin{aligned} a(z) &:= z^p + a_1 z^{p-1} + \cdots + a_p \,, \\ b(z) &:= z^q + b_{q-1} z^{q-1} + \cdots + b_0, \end{aligned}$$

Since DL_t does not exist in the usual sense, we interpret the differential equation (3.1) by means of its state-space representation, consisting of the *observation* and *state* equations,

$$V_t = \mathbf{b}' \mathbf{X}_t, \tag{3.2}$$

and

$$d\mathbf{X}_t - \mathbf{A}\mathbf{X}_t dt = \mathbf{e} \, dL_t, \tag{3.3}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \ \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}, \tag{3.4}$$

$b_q := 1$ and $b_j := 0$ for $j > q$. Every solution of (3.3) satisfies the relations

$$\mathbf{X}_t = e^{\mathbf{A}(t-s)} \mathbf{X}(s) + \int_s^t e^{\mathbf{A}(t-u)} \mathbf{e} \, dL_u, \ t > s,$$

where the integral is a special case of integration with respect to a semimartingale.

Brockwell and Lindner [17], Theorem 4.1, show that there is no loss of generality in assuming that $a(z)$ and $b(z)$ have no common factors and that, assuming L is non-deterministic, necessary and sufficient conditions for (3.2) and (3.3) to have a strictly stationary solution V are that $E\max(0, \log|L_1|)$ is finite and $a(z)$ is non-zero on the imaginary axis. In this case the strictly stationary solution is unique and is given by

$$V_t = \int_{-\infty}^{\infty} g(t-u)\, dL_u, \tag{3.5}$$

with

$$g(t) = \left(\sum_{\lambda:\Re\lambda<0} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} \mathbf{1}_{(0,\infty)}(t) - \sum_{\lambda:\Re\lambda>0} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} \mathbf{1}_{(-\infty,0)}(t) \right), \tag{3.6}$$

where the sums are over the distinct zeroes λ of the polynomial $a(z)$ and $\mu(\lambda)$ denotes the multiplicity of λ. The sum $\sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t}$ is the residue of $z \mapsto e^{zt} b(z)/a(z)$ at λ, i.e.

$$\sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} = \frac{1}{(\mu(\lambda)-1)!} \left[D_z^{\mu(\lambda)-1} \left((z-\lambda)^{\mu(\lambda)} e^{zt} b(z)/a(z) \right) \right]_{z=\lambda},$$

and D_z denotes differentiation with respect to z. (For a zero λ with $\mu(\lambda) = 1$ the last sum reduces to $b(\lambda)e^{\lambda t}/a'(\lambda)$.)

Remark 3.1. (**Causality**). The unique strictly stationary solution is causal if and only if $a(z)$ has no zeroes with positive real part, in which case the second sum in (3.6) disappears and g can be expressed as

$$g(t) = \begin{cases} \mathbf{b}' e^{At} = \frac{1}{2\pi i} \int_\rho \frac{b(z)}{a(z)} e^{tz} dz, & \text{if } t > 0, \\ 0, & \text{if } t \le 0. \end{cases} \tag{3.7}$$

where the subscript ρ indicates integration anticlockwise around a simple closed contour encircling the zeroes of $a(z)$ and contained in the open left half of the complex plane. If the zeroes all have multiplicity 1, we obtain the very simple representation,

$$V_t = \sum_{j=1}^{p} \frac{b(\lambda_j)}{a'(\lambda_j)} \int_{-\infty}^{t} e^{\lambda_j (t-u)} dL_u, \tag{3.8}$$

where a' denotes the derivative of a. From now on we shall restrict attention to causal CARMA processes. The term stationary will be used to indicate strict (as opposed to weak or covariance) stationarity, except when explicitly stated otherwise. □

Example 3.2. (**The stationary Ornstein-Uhlenbeck process**). In the case when $p = 1$ (so that q is necessarily zero) V is the CARMA(1,0) process, also written as CAR(1) (continuous-time autoregression of order 1) and widely known as the stationary Ornstein-Uhlenbeck process. In this case the dimension of the state-vector $\mathbf{X}_t$ is 1, $b(z) = 1$ and $a(z) = z - a_1$ where, for causality, $a_1 = \lambda_1 < 0$. Provided $E\max(0, \log|L_1|) < \infty$, $V_t = \mathbf{X}_t$ is the unique stationary solution of the equation,

$$dV_t - a_1 V_t dt = dL_t. \tag{3.9}$$

From (3.8) we immediately find that

$$V_t = \int_{-\infty}^{t} e^{\lambda_1(t-u)} dL_u \tag{3.10}$$

If L is a subordinator, i.e. a Lévy process with non-decreasing sample-paths, then inspection of (3.10) shows that the process V is non-negative. The subordinator-driven CAR(1) process is thus a potential model for any non-negative process such as the stochastic volatility considered later in Section 4. □

Example 3.3. (**The CARMA(2,1) process**). The CARMA(2,1) process with $a(z) = (z - \lambda_1)(z - \lambda_2)$, and $\lambda_1 \neq \lambda_2$, has the particularly simple structure,

$$V_t = \alpha_1 \int_{-\infty}^{t} e^{\lambda_1(t-u)} dL_u + \alpha_2 \int_{-\infty}^{t} e^{\lambda_2(t-u)} dL_u,$$

where

$$\alpha_i = \frac{b(\lambda_i)}{a'(\lambda_i)} = \frac{\lambda_i + b_0}{2\lambda_i + a_1}, \quad i = 1, 2.$$

The process is thus a sum of two dependent and possibly complex-valued CAR(1) processes. (Such a decomposition clearly extends to any CARMA(p, q) process for which $a(z)$ has distinct zeroes.) If L is a subordinator then, as in Example 3.2, V_t is non-negative provided the kernel $g(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t}$, $t \geq 0$, is non-negative. This is the case if and only if λ_1 and λ_2 are both real and $b_0 \geq \max(|\lambda_i|)$ (See Brockwell and Davis [15]. More general conditions for non-negativity of a CARMA(p, q) kernel are given by Tsai and Chan [42].) □

3.1 Second-order properties when $EL_1^2 < \infty$

If $EL_1^2 < \infty$, we define $\mu := EL_1$ and $\sigma^2 := \mathrm{Var}(L_1)$.

The causal CARMA process defined by (3.5) and (3.7) is then covariance stationary with mean $\mu b_0/a_p$ and autocovariance function which can be calculated as follows. From (3.7), noting that $g(t) = 0$ for $t < 0$, we see that the Fourier transform of g is

$$\tilde{g}(\omega) := \int_{\mathbb{R}} g(t)e^{i\omega t}dt = -\frac{1}{2\pi i}\int_{\rho}\frac{b(z)}{a(z)}\frac{1}{z+i\omega}dz = \frac{b(-i\omega)}{a(-i\omega)}, \quad \omega \in \mathbb{R}.$$

Since the autocovariance function $\gamma_V(\cdot)$ is the convolution of $\sigma g(\cdot)$ and $\sigma g(-\cdot)$, its Fourier transform is given by

$$\tilde{\gamma}_V(\omega) = \sigma^2\tilde{g}(\omega)\tilde{g}(-\omega) = \sigma^2\left|\frac{b(i\omega)}{a(i\omega)}\right|^2, \quad \omega \in \mathbb{R}.$$

The spectral density of V is the inverse Fourier transform of γ_V. Thus

$$f_V(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega h}\gamma_V(h)dh = \frac{1}{2\pi}\tilde{\gamma}_V(-\omega) = \frac{\sigma^2}{2\pi}\left|\frac{b(i\omega)}{a(i\omega)}\right|^2, \quad \omega \in \mathbb{R}.$$

Substituting this expression into the relation

$$\gamma_V(h) = \int_{\mathbb{R}} e^{i\omega h}f_V(\omega)d\omega, \quad h \in \mathbb{R},$$

and changing the variable of integration from ω to $z = i\omega$ gives,

$$\gamma_V(h) = \frac{\sigma^2}{2\pi i}\int_{\rho}\frac{b(z)b(-z)}{a(z)a(-z)}e^{|h|z}dz = \sigma^2\sum_{\lambda} Res_{z=\lambda}\left(e^{z|h|}b(z)b(-z)/(a(z)a(-z))\right),$$

where the sum is again over the distinct zeroes, λ of $a(z)$. This gives the general expression,

$$\gamma_V(h) = \sigma^2\sum_{\lambda}\frac{1}{(\mu(\lambda)-1)!}\left[D_z^{\mu(\lambda)-1}\frac{(z-\lambda)^m e^{z|h|}b(z)b(-z)}{a(z)a(-z)}\right]_{z=\lambda}, \tag{3.11}$$

where $\mu(\lambda)$ is the multiplicity of λ. In the case when the roots are distinct, equation (3.11) simplifies to

$$\gamma_V(h) = \sigma^2\sum_{\lambda:a(\lambda)=0}\frac{e^{\lambda|h|}b(\lambda)b(-\lambda)}{a'(\lambda)a(-\lambda)}. \tag{3.12}$$

Example 3.4. (**The second-order CAR(1) process**). If $EL_1^2 < \infty$ then, by (3.12), the CAR(1) process defined in Example 3.2 has the autocovariance function

$$\gamma_V(h) = \frac{\sigma^2}{2|\lambda_1|} e^{\lambda_1 |h|} \tag{3.13}$$

and autocorrelation function $\rho_V(h) = e^{\lambda_1 |h|}$. □

Example 3.5. (**The second-order CARMA(2,1) process**). If $EL_1^2 < \infty$ then, by (3.12), the CARMA(2,1) process defined in Example 3.3 with $\lambda_1 \neq \lambda_2$ has the autocovariance function,

$$\gamma_V(h) = \frac{\sigma^2}{2\lambda_1\lambda_2(\lambda_1^2 - \lambda_2^2)} \left[\lambda_2(b_0^2 - \lambda_1^2)e^{\lambda_1|h|} - \lambda_1(b_0^2 - \lambda_2^2)e^{\lambda_2|h|}\right].$$

This is a much broader class of functions than those in Example 3.4, allowing the approximation of a wider class of sample autocovariances than is possible when attention is restricted to CAR(1) models. □

4 A continuous-time stochastic volatility model

Let λ be strictly negative and let L be a subordinator. Then the spot volatility process V in the stochastic volatility model of Barndorff-Nielsen and Shephard [5] is defined, apart from a change of time scale, as the strictly stationary solution of the equation

$$dV_t = \lambda V_t \, dt + dL_t, \tag{4.1}$$

i.e. as a subordinator-driven CAR(1) process with driving Lévy process L and coefficient λ. Then, by (3.10), V_t is positive for all $t \in \mathbb{R}$. If G_t denotes the logarithm of the asset price at time t, then the process $(G_t)_{t\geq 0}$ is assumed to satisfy the stochastic differential equation,

$$dG_t = (m + bV_t)\, dt + \sqrt{V_t}\, dW_t, \tag{4.2}$$

where m and b are constants, and $(W_t)_{t\geq 0}$ is a standard Brownian motion, independent of L.

Notation. The term volatility is sometimes used to refer to V_t and sometimes to $\sqrt{V_t}$. We shall refer to V_t as the (spot) volatility at time t and to integrals of V_t over time intervals as integrated volatility. □

If $EL_1^2 < \infty$ and $\mathrm{Var}(L_1) = \sigma^2$, the autocovariance function of V is, by (3.13), the exponentially decaying function,

$$\mathrm{Cov}(V_{t+h}, V_t) = \sigma^2 e^{\lambda|h|}/(2|\lambda|).$$

If additionally $m = b = 0$, then (see Barndorff-Nielsen and Shephard [5], Section 4) non-overlapping increments of G of length $r > 0$ are uncorrelated, i.e.

$$\mathrm{Cov}(G_t - G_{t-r}, G_{t+h} - G_{t+h-r}) = 0, \quad t, h \geq r,$$

while, if $EL_1^4 < \infty$, the squared increments are correlated with the autocovariance function,

$$\mathrm{Cov}((G_t - G_{t-r})^2, (G_{t+h} - G_{t+h+r})^2) = C_r e^{-\lambda h}$$

for strictly positive integer multiples h of $r > 0$, where $C_r > 0$ is some constant. The process $((G_{rh} - G_{r(h-1)})^2)_{h\in\mathbb{N}}$ thus has the autocovariance structure of an ARMA(1,1) process. The fact that the increments of the log-price process are uncorrelated while its squares are not is one of the important *stylized features* of financial time series. The tail behaviour of the squared volatility process depends on the tail behaviour of the driving Lévy process. In particular, it can be seen that V_t has Pareto tails, i.e. that $P(V_t > x)$ behaves asymptotically as a constant times $x^{-\alpha}$ for some $\alpha > 0$ as $x \to \infty$, if and only if L_1 has Pareto tails with the same index α (see Fasen et al. [24]; the converse follows from the monotone density theorem for regularly varying functions, see e.g. Theorem 1.7.2 in Bingham et al. [11]).

5 Integrated CARMA processes and spot volatility modelling

In the stochastic volatility model (4.1) and (4.2), the spot volatility V_t is represented by a stationary Lévy-driven Ornstein-Uhlenbeck process. This has the shortcoming that its autocorrelation function is necessarily a decreasing exponential function. Spot volatility is not an observable quantity, however the *integrated volatility sequence*

$$I_n^\Delta = \int_{(n-1)\Delta}^{n\Delta} V_t \, dt, \ n = 1, 2, \ldots, \tag{5.1}$$

over successive periods of length Δ can be well estimated in the context of the model (4.1) and (4.2) by the so-called *realized volatility sequence*,

$$R_n = \sum_{j=1}^{k} d_{n,j}^2, \tag{5.2}$$

where

$$d_{n,j} = (G_{(n-1+j/k)\Delta} - G_{(n-1+(j-1)/k)\Delta})^2,$$

and k is large. Typically Δ denotes a single trading day and k is such that Δ/k is a five-minute interval. An excellent discussion of realized volatility can be found in the article of Andersen and Benzoni [3]. Figure 1 shows the realized daily volatility (kindly supplied by Viktor Todorov) of the Deutsche Mark/US dollar exchange rate from December 1, 1986, through June 30, 1999. (See Andersen et al. [1] for a discussion of the series on which this realized volatility was based.)

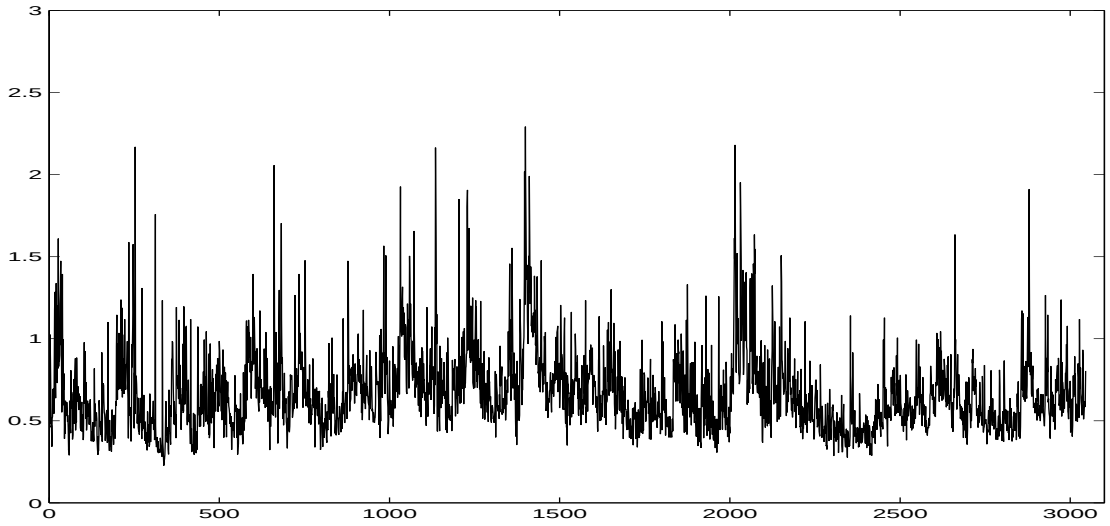

Figure 1: The realized daily volatility of the DM/US$ exchange rate, December 1, 1986, through June 30, 1999 .

The sample autocorrelation function of the series is shown in Figure 2. Under the CAR(1) model (4.1) for the spot volatility V_t, it has been shown by Barndorff-Nielsen

and Shephard [5] that the daily integrated volatility is an ARMA(1,1) process so that its autocorrelation function at lags greater than zero is a decreasing exponential function. It is clear from Figure 2 that a better fit should be achievable by modelling V as a higher-order CARMA process. Todorov and Tauchen [40]), Todorov [41] and Brockwell, Davis and Yang [16] fitted CARMA(2,1) models to the *daily realized volatility*. In this section we take a different point of view, the goal being to replace (4.1) by a CARMA model for the *spot volatility* V in such a way that the corresponding integrated volatility sequence (5.1) adequately reflects the properties of the observed realized volatility sequence (5.2).

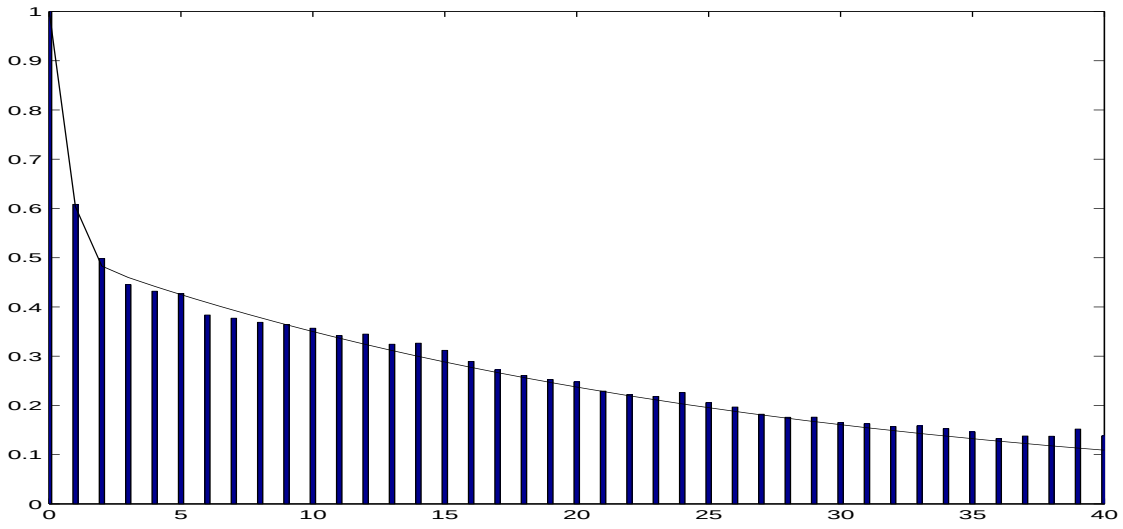

Figure 2: The vertical bars represent the sample autocorrelation function of the daily realized volatility of the DM/US$ exchange rate shown in Figure 1. The line graph is the autocorrelation function of the integrated volatility corresponding to a CARMA(2,1) model for spot volatility, estimated as described in Example 5.1.

If $EL_1^2 < \infty$ and V is the L-driven CARMA(p,q) process with autoregressive and moving average polynomials $a(z)$ and $b(z)$ respectively, Brockwell and Lindner [18] show that the integrated volatility sequence I^Δ is a weakly stationary solution of the difference equations,

$$\phi(B)I_n^\Delta = \theta(B)\epsilon_n,$$

where $(\epsilon_n)_{n\in\mathbb{Z}}$ is an uncorrelated constant variance sequence, B is the backward shift

operator (i.e. $B^j Y_n := Y_{n-j},$ for all j and $n \in \mathbb{Z}$) and $\phi(z)$ is the polynomial,

$$\phi(z) := \prod_{\lambda} (1 - e^{\lambda\Delta} z)^{\mu(\lambda)},$$

where the product is over the distinct zeroes λ of $a(z)$ and $\mu(\lambda)$ denotes the multiplicity of λ. The polynomial $\theta(z)$ has the form,

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_p z^p,$$

with coefficients $\theta_1, \ldots, \theta_p$ which can be determined from $a(z)$ and $b(z)$ and chosen to have no zeroes in the interior of the unit circle. For any $a(z)$ and $b(z)$, the corresponding ARMA polynomials $\phi(z)$ and $\theta(z)$ for the ARMA process I^Δ can therefore be determined and hence the minimum mean-squared-error one-step linear predictors of the sequence I^Δ. Numerical minimization of the sum of squares of these one-step errors with respect to the coefficients of the polynomials $a(z)$ and $b(z)$ gives least squares estimates of the CARMA coefficients for the spot volatility process V.

Example 5.1. To illustrate the procedure we consider the daily realized volatility in Figure 1. It is clear that a good match between the sample autocorrelation function in Figure 2 for lags greater than zero and a single exponential function (as would be derived from an Ornstein-Uhlenbeck model for spot volatility) is not possible. We therefore try a CARMA(2,1) model for the spot volatility. Measuring the spot volatility in units of volatility per day, the realized volatility series corresponds to volatility integrated over time intervals of length 1, i.e I^Δ with $\Delta = 1$.

A simple initial guess at appropriate values of the coefficients can be obtained by attempting to match the autocorrelation function of I^Δ with the sample autocorrelation of the realized volatility V^Δ at selected lags. If, for example, we minimize the sum of squared differences at lags 1, 2, 10, 20 and 40, we obtain the preliminary spot-volatility model,

$$(D^2 + 3.09054D + .10983)V_t = (.23302 + D)DL_t,$$

with corresponding $\lambda_1 = -.035956$ and $\lambda_2 = -3.05458$.

Using these coefficients as initial values, the numerical minimization of the prediction sum of squares leads to the least-squares model,

$$(D^2 + 3.07141D + .11793)V_t = (.23938 + D)DL_t, \tag{5.3}$$

with corresponding $\lambda_1 = -.038890$ and $\lambda_2 = -3.02152$. The autocorrelation function of the daily integrated volatility corresponding to this model is plotted as the line graph in Figure 2.

It remains to identify a subordinator L which yields daily integrated volatilities compatible with the realized daily volatility series shown in Figure 1. This was done by trying several subordinators, each with $EL_1 = .3291$ and $\mathrm{Var}L_1 = .3954$ to match the mean and variance of the realized volatility series, simulating sample paths of the corresponding CARMA(2,1) processes defined by (5.3), integrating the sample-paths over successive days and comparing the empirical cumulative distribution functions and kernel density estimates of the realized volatility series with those of the integrated volatilities calculated from the models. The results are shown in Figure 3.

The top graphs were generated by simulating the CARMA(2,1) process (5.3) driven by a compound Poisson subordinator with exponentially distributed jumps. The mean jump rate of the process was .5478 and the mean jump size was .6008. The simulation of the CARMA process is greatly simplified by the decomposition in Example 3 which reduces the simulation to that of two Ornstein-Uhlenbeck processes with the same driving subordinator. In fact from the simulated jump-times and jump-sizes the complete sample-path can be constructed and the daily integrals easily computed. The same is true for compound-Poisson-driven CARMA processes of any order as long as the zeroes of $a(z)$ are distinct.

The middle graphs are derived from the spot volatility process (5.3) with inverse Gaussian subordinator having $EL(1) = .3291$ and $\mathrm{Var}L(1) = .3954$. Simulation in this case was carried out by using an Euler approximation to generate values of the spot volatility at intervals of .01 days and integrating numerically to get 40,000 daily integrated volatility values.

The bottom graphs were derived in the same way, using a gamma subordinator with $EL(1) = .3291$ and $\mathrm{Var}L(1) = .3954$. The empirical cdf and kernel density estimates were again based on 40,000 daily integrated volatility values.

The reasonable fits by all three subordinators suggest that the distribution of daily integrated volatilities is rather insensitive to the distribution of L_1, however in the case of the gamma subordinator the empirical and simulated distributions are virtually indistinguishable. □

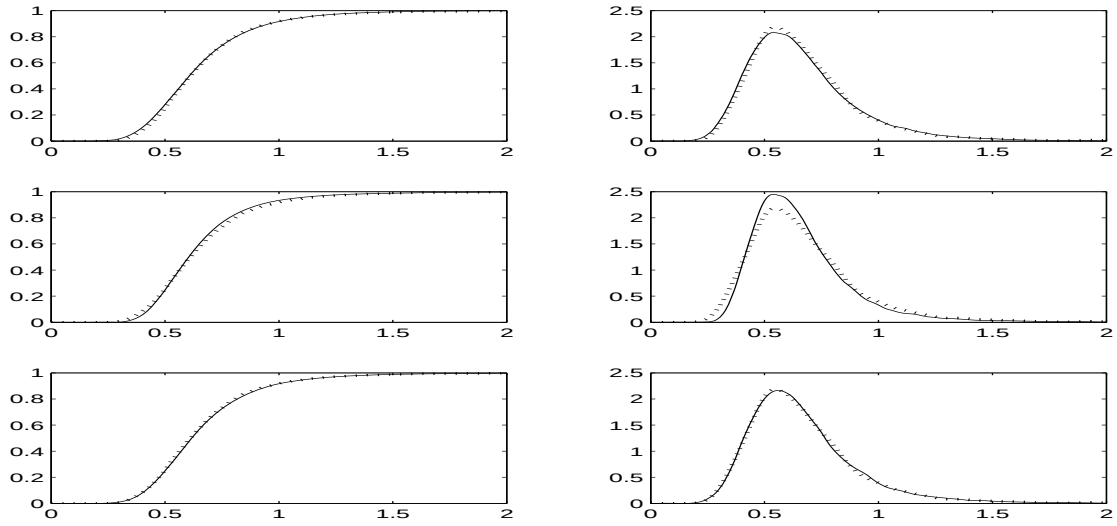

Figure 3: The empirical cdf (left) and kernel density estimate (right) of the daily realized volatility of the DM/US$ exchange rate are shown as dotted lines. The solid lines are the corresponding graphs for daily integrated volatility of three subordinator-driven CARMA(2,1) spot volatility processes. For details see Example 5.1.

6 Generalized Ornstein-Uhlenbeck processes

The Lévy-driven CARMA processes can be regarded as a higher-order generalization of the Ornstein-Uhlenbeck process. From (3.9) the Lévy-driven Ornstein-Uhlenbeck process satisfies the equation,

$$dV_t = \lambda_1 V_{t-} dt + dL_t, \ t \geq 0.$$

Another way in which to extend the class of Ornstein–Uhlenbeck processes is to replace the deterministic function $t \mapsto \lambda_1 t$ in this equation by a second Lévy process $(U_t)_{t\geq 0}$. This leads to the stochastic differential equation

$$dV_t = V_{t-}\, dU_t + dL_t, \quad t \geq 0, \tag{6.1}$$

or equivalently,

$$V_t = V_0 + \int_0^t V_{s-}\, dU_s + L_t, \quad t \geq 0,$$

where V_0 is a starting random variable and (U, L) is a bivariate Lévy process. We shall always assume that neither U nor L is the zero process. This stochastic differential equation was considered and solved by Yoeurp and Yor [43] (cf. Protter [36], Exercise V.27), see also Behme et al. [10]. In particular, it is shown that if U has no jumps of size -1, i.e. if $\nu_U(\{-1\}) = 0$, then the unique solution to (6.1) is given by

$$V_t = \mathcal{E}(U)_t \left(V_0 + \int_0^t [\mathcal{E}(U)_{s-}]^{-1} \, d\eta_s \right), \quad t \geq 0. \tag{6.2}$$

Here, $\mathcal{E}(U)$ denotes the stochastic exponential of U, given by

$$\mathcal{E}(U)_t = e^{U_t - t\sigma_U^2/2} \prod_{0<s\leq t} (1 + \Delta U_s) e^{-\Delta U_s}, \quad t \geq 0,$$

and η is given by

$$\eta_t = L_t - \sum_{0<s\leq\leq t} \frac{\Delta U_s \Delta L_s}{1 + \Delta U_s} - t\sigma_{U,L}, \quad t \geq 0,$$

where σ_U^2 and $(\sigma_{U,L})$ denote the $(1,1)$ and $(1,2)$-elements of the Gaussian variance $A_{(U,L)}$ of (U, L), respectively.

When U has no jumps of size less than or equal to -1, i.e. when $\nu_U((-\infty, -1]) = 0$, then $\mathcal{E}(U)_t$ is strictly positive and we can define

$$\xi_t := -\log \mathcal{E}(U)_t = -U_t + \sigma_U^2 t/2 + \sum_{0<s\leq t} (\Delta U_s - \log(1 + \Delta U_s)), \quad t \geq 0.$$

Then $(\xi, \eta) = (\xi_t, \eta_t)_{t\geq 0}$ is again a bivariate Lévy process and the process in (6.2) can be written as

$$V_t = e^{-\xi_t} \left(V_0 + \int_0^t e^{\xi_{s-}} \, d\eta_s \right), \quad t \geq 0. \tag{6.3}$$

If V_0 is additionally independent of (ξ, η) (equivalently, of (U, L)), then the process given by (6.3) is called a *generalized Ornstein–Uhlenbeck process*, driven by (ξ, η). This terminology is due to de Haan and Karandikar [26] and Carmona et al. [19], who studied various properties of these processes. The process (U, L) can be recovered from (ξ, η) by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} = \begin{pmatrix} -\xi_t + \sum_{0<s\leq t} \left(e^{-\Delta\xi_s} - 1 + \Delta\xi_s \right) + t\,\sigma_\xi^2/2 \\ \eta_t + \sum_{0<s\leq t} (e^{-\Delta\xi_s} - 1)\Delta\eta_s - t\,\sigma_{\xi,\eta} \end{pmatrix}, \quad t \geq 0.$$

Obviously, if U and L are independent, then so are ξ and η (and conversely), in which case $\eta_t = L_t$.

It is clear that if $\xi_t = -U_t = -\lambda_1 t$, $t \geq 0$, then a generalized Ornstein–Uhlenbeck process reduces to the Lévy driven Ornstein–Uhlenbeck process defined by (3.9).

We have already seen that generalized Ornstein–Uhlenbeck processes arise as Lévy-driven Ornstein–Uhlenbeck processes in the stochastic volatility model of Barndorff–Nielsen and Shephard [5]. In Section 7 we shall see that generalized Ornstein–Uhlenbeck processes also arise as volatility processes of continuous-time GARCH(1,1) processes, when ξ is deterministic and η random in contrast to the situation of the Lévy-driven Ornstein–Uhlenbeck process. In general, since generalized Ornstein–Uhlenbeck processes are the natural continuous time analogues of AR(1) processes with random i.i.d. coefficients (cf. de Haan and Karandikar [26]), a non-negative generalized Ornstein–Uhlenbeck process may serve as a stochastic volatility model and hence has potential applications in finance, even when not restricted to the continuous time GARCH situations. The generalized Ornstein–Uhlenbeck process is non-negative if $V_0 \geq 0$ and η is a subordinator.

Another branch of finance in which generalized Ornstein–Uhlenbeck processes make an appearance is insurance mathematics, specifically in the risk model of Paulsen [35]. Here, V_t denotes the capital of an insurance company, L_t describes the premium minus the claim process, and U describes the behavior of a financial market in which the capital of the insurance company is invested. See Paulsen [35] for details.

For using a generalized Ornstein–Uhlenbeck process as a volatility model, it is interesting to know for which bivariate Lévy processes (ξ, η) there exists a starting random variable V_0, independent of (ξ, η), such that the corresponding generalized Ornstein–Uhlenbeck process becomes strictly stationary. A complete characterization of this was obtained by Lindner and Maller [31]. Accordingly, a strictly stationary solution exists if and only if there is $k \in \mathbb{R} \setminus \{0\}$ such that $e^{\xi} = \mathcal{E}(\eta/k)$, in which case $V_t = k$ for all $t \geq 0$, or if the integral $\int_0^t e^{-\xi_{s-}}\, dL_s$ converges almost surely as $t \to \infty$, in which case the marginal stationary distribution is given by the distribution of

$$\int_0^\infty e^{-\xi_{s-}}\, dL_s. \tag{6.4}$$

A necessary and sufficient condition for the integral in (6.4) to converge almost surely absolutely has been given by Erickson and Maller [23]. A sufficient condition for the convergence of the integral is that $E \log^+ |L_1| < \infty$ and that ξ_t converges almost surely to $+\infty$, the latter being implied by $E\xi_1 > 0$ (cf. de Haan and Karandikar [26] and Lindner and Maller [31]). An extension of the characterization of stationary solutions of generalized Ornstein–Uhlenbeck processes to solutions of (6.1), when U is allowed to have jumps of size less than or equal to -1 and V_0 is allowed also to be dependent of (U, L), hence allowing also non-causal solutions, has been given by Behme et al. [10].

The autocorrelation structure of a generalized Ornstein–Uhlenbeck is always of exponential form. More precisely, if U and L are such that $\nu_U((-\infty,-1]) = 0$, $EU_1^2, EL_1^2 < \infty$ and $E\mathcal{E}(U)_1^2 = Ee^{-2\xi_1} < 1$, then $EU_1 < 0$ and a stationary version with finite second moment of the generalized Ornstein-Uhlenbeck process exists, the mean of which is given by $EV_0 = -(EU_1)^{-1}EL_1$ and the autocovariance function by

$$\mathrm{Cov}(V_t, V_{t+h}) = \frac{E(U_1EL_1 - L_1EU_1)^2}{(EU_1)^2|2EU_1 + \mathrm{Var}U_1|}\, e^{hEU_1}, \quad t, h \geq 0,$$

see Behme [7, 8].

Another important feature of generalized Ornstein–Uhlenbeck processes is that they allow the stationary solution to have Pareto tails for a wide variety of situations, even if η does not have heavy tails. This follows from the results of Kesten [28] and Goldie [25] on the tail behavior of solutions of random recurrence equations; see Behme [7, 8] and Lindner and Maller [31] for details. Finally, we remark that multivariate extensions of generalized Ornstein–Uhlenbeck processes have been recently obtained by Behme [8]; see also Behme and Lindner [9].

7 Continuous-time GARCH processes

Among the most prominent discrete time models for financial time series are the ARCH and GARCH processes of Engle [22] and Bollerslev [12]. Given an i.i.d. sequence $(\varepsilon_n)_{n\in\mathbb{N}_0}$ and constants $\beta > 0$, $\lambda_1, \ldots, \lambda_q \geq 0$ and $\delta_1, \ldots, \delta_p \geq 0$ with $q \in \mathbb{N}$ and $p \in \mathbb{N}_0$ and $\lambda_q > 0$, a GARCH(q,p) process $(Y_n)_{n\in\mathbb{N}_0}$ with volatility process $(V_n)_{n\in\mathbb{N}_0}$ is given by

$$Y_n = \sqrt{V_n}\,\varepsilon_n, \quad n \in \mathbb{N}_0, \tag{7.1}$$

$$V_n = \beta + \sum_{i=1}^{q} \lambda_i Y_{n-i}^2 + \sum_{j=1}^{p} \delta_j V_{n-j}, \quad n \geq \max\{p,q\}, \tag{7.2}$$

with V_n independent of $(\varepsilon_{n+h})_{h\in\mathbb{N}_0}$ and non-negative for every $n \in \mathbb{N}_0$. For $p = 0$ the process is called ARCH(q) process.

Continuous time diffusion limits have been obtained by Nelson [34] for the GARCH (1,1) process and by Duan [21] for the GARCH(q,p) process. By considering GARCH processes on fine grids $h\mathbb{N}_0$ and rescaling the parameters appropriately as $h \downarrow 0$, Nelson

obtained the following diffusion limit $(G_t, V_t)_{t\geq 0}$ given by

$$dG_t = \sqrt{V_t}\, dB_t^{(1)}, \quad t \geq 0, \tag{7.3}$$

$$dV_t = (\omega - \theta V_t)\, dt + \alpha V_t\, dB_t^{(2)}, \quad t \geq 0, \tag{7.4}$$

where $B^{(1)}$ and $B^{(2)}$ are two independent Brownian motions and $\theta \in \mathbb{R}$, $\omega \geq 0$ and $\alpha > 0$ are parameters. In particular, the volatility process determined by (7.4) is a generalized Ornstein–Uhlenbeck process driven by $(\xi_t, \eta_t) = (-\alpha B_t^{(2)} + (\theta + \alpha^2/2)t, \omega t)$. It should be observed that the diffusion limit of Nelsen has two independent sources of randomness, namely $B^{(1)}$ and $B^{(2)}$, while the GARCH(1,1) process defined by (7.1) and (7.2) is driven by a single noise process $(\varepsilon_n)_{n\in\mathbb{N}_0}$. This motivated Klüppelberg et al. [29] to construct a continuous-time GARCH(1,1) process driven by a single Lévy process, called COGARCH(1,1). Given a driving Lévy process $M = (M_t)_{t\geq 0}$ with nonzero Lévy measure, independent of a starting random variable $V_0 \geq 0$, and constants $\beta, \delta > 0$ and $\lambda \geq 0$, they define the *COGARCH(1,1) process* $(G_t)_{t\geq 0}$ with *volatility process* $(V_t)_{t\geq 0}$ by

$$G_0 = 0, \quad dG_t = \sqrt{V_{t-}}\, dM_t, \quad t \geq 0,$$

where

$$V_t = \left(\beta \int_0^t e^{\xi_{s-}}\, ds + V_0\right) e^{-\xi_t}, \quad t \geq 0,$$

and $\xi = (\xi_t)_{t\geq 0}$ is defined by

$$\xi_t := -t \log \delta - \sum_{0<s\leq t} \log(1 + \lambda \delta^{-1} (\Delta M_s)^2), \quad t \geq 0.$$

Then ξ is again a Lévy process and V is a generalized Ornstein–Uhlenbeck process, driven by the bivariate Lévy process $(\xi_t, \beta t)_{t\geq 0}$. The corresponding processes (U, L) in the differential equation (6.1) are given by $U_t = t \log \delta + \lambda \delta^{-1} \sum_{0<s\leq t} (\Delta M_s)^2$ and $L_t = \beta t$, i.e.

$$dV_t = V_{t-} d(t \log \delta + \lambda \delta^{-1} [M, M]_t^{(d)}) + \beta dt, \quad t \geq 0,$$

where $[M, M]_t^{(d)} = \sum_{0<s\leq t} (\Delta M_s)^2$ denotes the discrete part of the quadratic variation of M. A multivariate extension of the COGARCH(1,1) process has been obtained by Stelzer [39].

It has been shown in Klüppelberg et al. [29] that a stationary volatility process of the COGARCH(1,1) equations exists if and only if

$$\int_{\mathbb{R}} \log(1 + \lambda \delta^{-1} x^2)\, \nu_M(dx) < -\log \delta, \tag{7.5}$$

which in particular requires M to have finite log-moment and $\delta < 1$. The second moment structure of V_t can be obtained from those of generalized Ornstein-Uhlenbeck processes. More precisely, under the condition (7.5) and defining

$$\Psi_\xi(\kappa) := \log E e^{-\kappa\xi_1} = \kappa \log \delta + \int_{\mathbb{R}} \left((1 + \lambda\delta^{-1}y^2)^\kappa - 1\right) \nu_M(dy) \in (-\infty, \infty]$$

for $\kappa > 0$, the stationary version satisfies for $k \in \mathbb{N}$ that $EV_0^k < \infty$ if and only if $EM_1^{2k} < \infty$ and $\Psi_\xi(k) < 0$, in which case $\Psi_\xi(l) < 0$ for all $l \in \{1, \ldots, k\}$ and

$$EV_0^k = k!\beta^k \prod_{l=1}^{k} (-\Psi_\xi(l))^{-1}.$$

Further, if $EM_1^4 < \infty$ and $\Psi_\xi(2) < 0$, then

$$\mathrm{Cov}(V_t, V_{t+h}) = \beta^2 \left(2\Psi_\xi^{-1}(1)\Psi_\xi^{-1}(2) - \Psi_\xi^{-2}(1)\right) e^{-h|\Psi_\xi(1)|}, \quad t, h \geq 0.$$

A detailed proof is given in Klüppelberg et al. [29].

As is the case for the volatility model of Barndorff-Nielsen and Shephard considered in Section 4, under certain assumptions, non-overlapping increments of the stationary COGARCH(1,1) process G are uncorrelated, while the autocovariance function of $((G_{rh} - G_{r(h-1)})^2)_{h\in\mathbb{N}}$ is that of an ARMA(1,1) process for any $r > 0$. More precisely, restricting to $r = 1$ for simplicity, if the driving Lévy process $M = (M_t)_{t\geq 0}$ satisfies

$$EM_1 = 0, \quad \mathrm{Var}(M_1) = 1, \quad EM_1^4 < \infty, \quad \int_{\mathbb{R}} x^3 \, \nu_M(dx) = 0,$$

and if

$$\Psi_\xi(2) = 2\log\delta + \int_{\mathbb{R}} \left(\lambda^2\delta^{-2}y^4 + 2\lambda\delta^{-1}y^2\right) \nu_\xi(dy) < 0,$$

then the increments process $(Y_n)_{n\in\mathbb{N}}$ with $Y_n = G_n - G_{n-1}$ satisfies $EY_1^4 < \infty$ and

$$EY_1 = 0, \quad \mu := E(Y_1^2) = \frac{\beta}{|\Psi_\xi(1)|} \quad \text{and} \quad \mathrm{Cov}(Y_t, Y_{t+h}) = 0, \quad t, h \in \mathbb{N}.$$

Denoting

$$\varphi := \lambda\delta^{-1} \quad \text{and} \quad \tau := -\log\delta,$$

the autocorrelation function ρ of Y satisfies

$$\rho(h) = ke^{-h\rho}, \quad t, h \in \mathbb{N}, \tag{7.6}$$

where

$$p := |\Psi_\xi(1)|$$

and

$$k := \frac{\beta^2}{p^3\gamma(0)}(2\tau\varphi^{-1} + 2A_M - 1)\left(2|\Psi_\xi^{-1}(2)| - p^{-1}\right)\left(1 - e^{-p}\right)(e^p - 1).$$

An explicit expression for $\mathrm{Var}(Y_0)$ can also be obtained. Based on these expressions, Haug et al. [27] consider a generalized method of moment estimator for the parameters of the COGARCH(1,1) process, by replacing $E(Y_1^2)$, $\mathrm{Var}(Y_0)$ and $\log\rho(h)$ by their empirical counterparts and doing a regression for p and k in (7.6). Assuming that the Gaussian variance A_M is known (e.g. $A_M = 0$), solving the equations obtained for $\mu, \mathrm{Var}(Y_0), p, k$ in β, φ, τ and plugging the obtained estimators $\widehat{\mu}$, $\widehat{\mathrm{Var}(Y_0)}$, $\widehat{p}$ and $\widehat{k}$ into these equations gives generalized method of moment estimators $(\widehat{\beta}, \widehat{\varphi}, \widehat{\tau})$ for the parameters (β, φ, τ) and hence for (β, δ, λ) based on observations $G_0, G_1, G_2, \ldots, G_n$. Details can be found in Haug et al. [27].There it is also shown that the estimator is strongly consistent and under further moment assumptions, which require in particular a finite 8^{th} moment of Y, that the estimator is asymptotically normal.

Other estimation methods for the COGARCH(1,1) include the pseudo-maximum likelihood estimator of Maller et al. [32] and the Markov Chain Monte Carlo estimator of Müller [33]. Maller et al. [32] also fit the COGARCH(1,1) model to the ASX200 index of the Australian Stock exchange.

Finally, let us introduce the COGARCH(q,p) processes of Brockwell et al. [14]. From (7.1) and (7.2) we see that the volatility (V_n) of a GARCH(q,p) process can be seen as a "self-exciting" ARMA$(p, q-1)$ process driven by $(V_{n-1}\varepsilon_{n-1}^2)$ together with the "mean correction" β. This motivates to define the volatility process $(V_t)_{t\geq 0}$ of a continuous-time GARCH(q,p) process as a "self-exciting mean corrected" CARMA$(p, q-1)$ process driven by an appropriate noise term. Since in discrete time, the driving noise is defined through the increments of the process $(\sum_{i=0}^{n-1} V_i\varepsilon_i^2)_{n\in\mathbb{N}}$, in continuous time this suggests the use of

$$R_t = \sum_{0<s\leq t} V_{s-}(\Delta M_s)^2 = \int_0^t V_{s-}d[M,M]_s^{(d)}, \quad t \geq 0,$$

as driving noise for the CARMA equations. More pecisely, let $M = (M_t)_{t\geq 0}$ be a Lévy process with nonzero Lévy measure. With $p, q \in \mathbb{N}$ such that $q \leq p$, $a_1, \ldots, a_p, b_0, \ldots, b_{p-1} \in \mathbb{R}, \beta > 0, a_p \neq 0, b_{q-1} \neq 0$ and $b_q = \ldots = b_{p-1} = 0$, define the $p \times p$-matrix $\mathbf{A}$ and the vectors $\mathbf{b}, \mathbf{e} \in \mathbb{C}^p$ as in (3.4). Define the volatility process $(V_t)_{t\geq 0}$ with parameters $\mathbf{A}$, $\mathbf{b}$, β and driving Lévy process M by

$$V_t = \beta + \mathbf{b}'\mathbf{X}_t, \quad t \geq 0,$$

where the state process $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ is the unique solution of the stochastic differential equation

$$d\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-}\,dt + \mathbf{e}V_{t-}\,d[M,M]_t^{(d)} = \mathbf{A}\mathbf{X}_{t-}\,dt + \mathbf{e}(\beta + \mathbf{b}'\mathbf{X}_{t-})\,d[M,M]_t^{(d)},$$

with initial value $\mathbf{X}_0$, independent of $(M_t)_{t\geq 0}$. If the process $(V_t)_{t\geq 0}$ is non-negative almost surely, then $G = (G_t)_{t\geq 0}$, defined by

$$G_0 = 0, \quad dG_t = \sqrt{V_{t-}}\,dM_t,$$

is a *COGARCH*(q,p) *process* with parameters $\mathbf{A}$, $\mathbf{b}$, β and driving Lévy process M.

It can be shown that for $p = q = 1$ this definition is equivalent to the definition of the COGARCH(1,1) process given before. Brockwell et al. [14] give sufficient conditions for the existence of a strictly stationary solution $(V_t)_{t\geq 0}$ and its positivity, and show that $(V_t)_{t\geq 0}$ has the same autocorrelation structure as a CARMA$(p, q-1)$ process. Hence the COGARCH(q,p) process allows a more flexible autocorrelation structure than the COGARCH(1,1) process. Under suitable conditions, which among others require M_1 to have expectation zero, it is further shown that non-overlapping increments of G are uncorrelated, while their squares are not. More precisely,

$$\mathrm{Cov}((G_t - G_{t-r})^2, (G_{t+h} - G_{t+h-r})^2) = \mathbf{b}'e^{(A+EM_1^2\mathbf{e}\mathbf{b}')h}H_r, \quad h \geq r > 0,$$

where $H_r \in \mathbb{C}^p$ is independent of h. In particular, the squared increments have a covariance structure similar to that of a CARMA process.

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