Decomposition of Riesz frames and wavelets into a finite union of linearly independent sets

Ole Christensen, Alexander M. Lindner *

Abstract

We characterize Riesz frames and prove that every Riesz frame is the union of a finite number of Riesz sequences. Furthermore, it is shown that for piecewise continuous wavelets with compact support, the associated regular wavelet systems can be decomposed into a finite number of linearly independent sets. Finally, for finite sets an equivalent condition for decomposition into a given number of linearly independent sets is presented.

1 Introduction

This paper is concerned with the decomposition of certain families of functions (more generally, vectors) into a finite number of linearly independent sets. Let $(H, < \cdot, \cdot >)$ denote a separable Hilbert space. A *frame* is a sequence $\{\varphi_i\}_{i \in I}$ of elements in H for which there exist positive constants A and B such that

$$A\|f\|^2 \le \sum_{i \in I} | < f, \varphi_i > |^2 \le B\|f\|^2 \quad \forall\, f \in H.$$

The constants A and B are called *frame bounds.*

Not every frame can be decomposed into a finite union of bases (see Example 2.1). We prove that every Riesz frame is the union of a finite

*Mathematics Subject Classification 2000: 42C15, 42C40 number of Riesz sequences (see below for the exact definitions). Our result is actually a consequence of a characterization of Riesz frames in terms of finite linearly independent subfamilies. This result is of independent interest, because it provides a criterion for $\{f_i\}_{i\in I}$ to be a Riesz frame that is easier to apply than the definition. Also, we shall give a geometric characteriziation of Riesz frames in terms of angles between certain Hilbert spaces.

Given a function $\psi \in L^2(\mathbb{R})$ and parameters $a > 1, b > 0$, the associated *wavelet system* is the family of functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ defined by

$$\psi_{j,k}(x) = a^{j/2}\psi(a^j x - kb),\ x \in \mathbb{R}. \tag{1}$$

Note that we use the word wavelet in a very general sense: we do not assume that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a basis or even a frame.

A family of the type $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ can not always be decomposed into a finite union of Riesz sequences. But we prove that if $\psi \neq 0$ is piecewise continuous and has compact support, then $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ can be decomposed into a finite union of linearly independent sets.

Finally we prove that a finite set $\{\varphi_i\}_{i\in I}$ of vectors can be decomposed into at most $E \in \mathbb{N}$ linearly independent families if and only if any subset $J \subseteq I$ satisfies

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_i : i \in J\}} \leq E.$$

2 Characterization of Riesz frames

A family $\{\varphi_i\}_{i\in I}$ of elements in H is by definition a *Riesz frame* [5] if it is complete and there are positive constants A and B such that for any subset J of I, $\{\varphi_i\}_{i\in J}$ is a frame for $\overline{span}\{\varphi_i\}_{i\in J}$ with frame bounds A and B. The common frame bounds A, B will be called *Riesz frame bounds*; note that the upper Riesz frame bound corresponds to the upper frame bound for $\{\varphi_i\}_{i\in I}$, while a lower frame bound for $\{\varphi_i\}_{i\in I}$ might not be a lower Riesz frame bound. For example, if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis, then $\{e_1, e_1, e_2, e_2, \dots\}$ is a frame with bounds $A = B = 2$; it is also a Riesz frame, but with bounds $A = 1, B = 2$. We also note that not every frame is a Riesz frame:

Example 2.1 Let again $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis, and let

$$\{\varphi_i\}_{i\in I} = \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\}.$$

Since the element $\frac{1}{\sqrt{k}}e_k, k \in \mathbb{N}$, appear k times in $\{\varphi_i\}_{i\in I}$, we have

$$\sum_{i\in I} |< f, \varphi_i >|^2 = \sum_{k=1}^{\infty} |< f, e_k >|^2 = ||f||^2, \ \forall f \in H,$$

i.e., $\{\varphi_i\}_{i\in I}$ is a frame for H with bounds $A = B = 1$. However, the subsequence $\{\frac{1}{\sqrt{k}}e_k\}_{k=1}^{\infty}$ is not a frame, as we see by taking $f = e_k, k \in \mathbb{N}$. Thus $\{\varphi_i\}_{i\in I}$ is not a Riesz frame. It is also clear that $\{\varphi_i\}_{i\in I}$ can not be decomposed into a finite union of bases.

In this section we give characterizations of Riesz frames. Our characterizations are in terms of certain properties for *finite* subfamilies of $\{\varphi_i\}_{i\in I}$. Recall that $\{\varphi_i\}_{i\in I}$ is a *Riesz sequence* if there are positive constants A and B such that

$$A\sum_{i\in J} |c_i|^2 \leq \left\|\sum_{i\in J} c_i\varphi_i\right\|^2 \leq B\sum_{i\in J} |c_i|^2$$

for all finite subsets J of I and all scalar sequences $(c_i)_{i\in J}$. The constants A and B are called *bounds* of the Riesz sequence. If furthermore $\{\varphi_i\}_{i\in I}$ is complete in H, then $\{\varphi_i\}_{i\in I}$ is a *Riesz basis.* Also, $\{\varphi_i\}_{i\in I}$ is called a *Bessel sequence* if at least the upper Riesz sequence condition is satisfied (or, equivalently, the upper frame condition is satisfied with the same constant B).

For infinite sequences there exist different concepts of "linear independence". We say that

(i) $\{\varphi_i\}_{i\in I}$ is *linearly independent* if every finite subset of $\{\varphi_i\}_{i\in I}$ is linearly independent

and that

(ii) $\{\varphi_i\}_{i\in I}$ is *ω-independent* if

$$\sum_{i\in I} c_i\varphi_i = 0 \Rightarrow c_i = 0, \ \forall i.$$

Riesz bases can be characterized in terms of those concepts:

Lemma 2.2 *Let $\{\varphi_i\}_{i\in I}$ be a frame for H. Let $\{I_n\}_{n\in\mathbb{N}}$ be a family of finite subsets of I such that*

$$I_1 \subset I_2 \subset I_3 \cdots \uparrow I.$$

Let A_n denote the optimal (i.e., the largest) lower frame bound for $\{\varphi_i\}_{i\in I_n}$ (as frame for its span). Then the following are equivalent:

(i) $\{\varphi_i\}_{i\in I}$ is a Riesz basis.

(ii) $\{\varphi_i\}_{i\in I}$ is ω-independent.

(iii) $\{\varphi_i\}_{i\in I}$ is linearly independent, and $\inf_n A_n > 0$.

For the proof we refer to [15] or [9]. A frame which is not a Riesz basis will be said to be *overcomplete*.

Lemma 2.2 implies that a linearly independent set is a Riesz frame if and only if it is a Riesz basis. ω-independence clearly implies linear independence. Theorem 2.4 will give a characterization of arbitrary Riesz frames in terms of linearly independent subsets. We begin with

Theorem 2.3 *The sequence $\{\varphi_i\}_{i\in I}$ is a Riesz frame for its closed linear span with bounds A and B if and only if (a) and (b) below hold:*

(a) $\{\varphi_j\}_{j\in J}$ is a Riesz sequence with bounds A and B whenever $J \subset I$ is a finite non-empty set such that $\{\varphi_j\}_{j\in J}$ is linearly independent.

(b) $\{\varphi_i\}_{i\in I}$ is a Bessel sequence with bound B.

Proof: From the definition of Riesz frames the "only if" implication of the above characterization is clear. For the converse, suppose that there is a positive constant A such that for any non-empty finite subset J of I for which $\{\varphi_j\}_{j\in J}$ is linearly independent it is already a Riesz sequence with lower bound A. Let $K \subset I$ be any non-empty set and let $f \in \operatorname{span}\{\varphi_j : j \in K\}$. By the definition of the span, this means that f has a representation as a linear combination of a *finite* set of vectors $\varphi_i,\ i \in K$. By deleting linearly dependent vectors if necessary, we find a finite subset J of K such that $f \in \text{span}\{\varphi_j : j \in J\}$ and $\{\varphi_j\}_{j\in J}$ is linearly independent. From the assumption it follows

$$A \cdot \|f\|^2 \leq \sum_{j\in J} | < f, \varphi_j > |^2 \leq \sum_{j\in K} | < f, \varphi_j > |^2 \leq B \cdot ||f||^2. \tag{2}$$

It is well known that if the frame condition holds on a subspace, then it also holds on the closure of the subspace. Thus (2) holds for all $f \in \overline{span}\{\varphi_j : j \in K\}$ and the conclusion follows. $\square$

Below we give another characterization of Riesz frames, substituting the condition of $\{\varphi_i\}_{i\in I}$ being a Bessel sequence by the condition that $\{\varphi_i\}_{i\in I}$ can be decomposed into a finite number of linearly independent sets:

Theorem 2.4 *The sequence $\{\varphi_i\}_{i\in I}$ is a Riesz frame for $\overline{span}\{\varphi_i\}_{i\in I}$ if and only if the following two conditions hold:*

(a) There are positive constants A', B' such that $\{\varphi_j\}_{j\in J}$ is an (A', B')-Riesz sequence whenever $J \subset I$ is a finite set such that $\{\varphi_j\}_{j\in J}$ is linearly independent.

(b) There is a finite partition of I into disjoint sets $D_1, \ldots, D_E$ such that each of the $\{\varphi_i\}_{i\in D_j}$ is linearly independent $(j = 1, \ldots, E)$.

Proof: Suppose first that $\{\varphi_i\}_{i\in I}$ is a Riesz frame for $H_1 := \overline{span}\{\varphi_i\}_{i\in I}$ Then (a) is clear. To prove (b), we use that there is a subset D_1 of I such that $\{\varphi_i\}_{i\in D_1}$ is a Riesz basis for H_1 (cf. [5]). Now consider the set $I \setminus D_1$. Then $\{\varphi_i\}_{i\in I\setminus D_1}$ is a Riesz frame for its closed linear span H_2. Thus it contains a Riesz basis $\{\varphi_i\}_{i\in D_2}$ for H_2. Continuing in this way we obtain a sequence of Hilbert spaces $H_1 \supset H_2 \supset H_3 \supset \ldots$ and a sequence of Riesz bases $\{\varphi_i\}_{i\in D_j}$ for H_j. By construction, the sets D_j are disjoint. We want to prove that this process stops after a finite number of steps, i.e., that for a certain positive integer E, $H_{E+1} = \{0\}$. Suppose there is some non-zero $f \in H_E$ for some $E \in \mathbb{N}$. We then have

$$f \in H_E \subset H_{E-1} \subset \ldots \subset H_1.$$

Denoting the Riesz frame bounds for $\{\varphi_i\}_{i\in I}$ by A, B, we have

$$A \cdot \|f\|^2 \leq \sum_{i\in D_j} | < f, \varphi_i > |^2 \quad \forall\, j = 1, \ldots, E.$$

Thus we obtain

$$A \cdot E \cdot \|f\|^2 \leq \sum_{j=1}^{E} \sum_{i \in D_j} | < f, \varphi_i > |^2 \leq \sum_{i \in I} | < f, \varphi_i > |^2 \leq B \cdot \|f\|^2.$$

Since $f \neq 0$, we have $E \leq B/A$. So $\{\varphi_i\}_{i \in I}$ can be partitioned into a finite number of linearly independent sets. This proves (b).

For the converse, suppose that conditions (a) and (b) are valid. Let J be any finite non-empty subset of I. Choose a partition of J into E disjoint subsets $J_1, \ldots, J_E$ such that $\{\varphi_i\}_{i \in D_j}$ is linearly independent for $j = 1, \ldots, E$. Then by (a), $\{\varphi_i\}_{i \in D_j}$ is a Bessel sequence with bound B'. Let $\{c_i\}_{i \in I} \in \ell^2(I)$. Then we have

$$\left\| \sum_{i \in J_j} c_i \varphi_i \right\|^2 \leq B' \sum_{i \in J_j} |c_i|^2$$

for $j = 1, \ldots, E$, and hence

$$\left\| \sum_{i \in J} c_i \varphi_i \right\| \leq \sum_{j=1}^{E} \left\| \sum_{i \in J_j} c_i \varphi_i \right\| \leq$$

$$\sqrt{B'} \sum_{j=1}^{E} \sqrt{\sum_{i \in J_j} |c_i|^2} \leq \sqrt{EB' \sum_{i \in J} |c_i|^2}.$$

By continuity, it thus follows that $\{\varphi_i\}_{i \in I}$ is a Bessel sequence with bound EB'. That it is a Riesz frame for its closed linear span now follows from Theorem 2.3. □

The proof of Theorem 2.4 has additionally shown:

Corollary 2.5 *Any Riesz frame $\{\varphi_i\}_{i \in I}$ can be partitioned into a finite number E of Riesz sequences $\{\varphi_i\}_{i \in D_j}$, $j = 1, \ldots, E$; if $\{\varphi_i\}_{i \in I}$ has the Riesz frame bounds A, B, we can choose $E \leq \frac{B}{A}$.*

Remark 2.6 The proof of Theorem 2.4 shows that condition (b) can be replaced by the following condition:
(b') There is an integer E such that any finite subset J of I can be partitioned into E disjoint sets J_j such that each of the $\{\varphi_i\}_{i \in J_j}$ is linearly independent.

Condition (a) in Theorems 2.3 and 2.4 is not a very geometric one. We can replace it by another condition, involving angles between closed subspaces; recall that the *angle* $\alpha(H_1, H_2)$ between two closed subspaces H_1 and H_2 of H is the unique number in the interval $[0, \pi/2]$ such that

$$\cos \alpha(H_1, H_2) = \sup\{| < f, g > | : f \in H_1, g \in H_2, \|f\| = \|g\| = 1\}.$$

A straightforward application of the results of [4], characterizing Riesz sequences in terms of angles, now gives:

Theorem 2.7 *For a sequence $\{\varphi_i\}_{i\in I} \subset H$ the following are equivalent:*

(a) There are positive constants A, B such that $\{\varphi_i\}_{i\in J}$ is an (A, B)-Riesz sequence whenever $J \subset I$ is a finite set such that $\{\varphi_i\}_{i\in J}$ is linearly independent.

(b) $\{\|\varphi_i\|\}_{i\in I}$ is bounded from above and from below by positive constants and there is some positive constant a such that for any two finite non-empty subsets M and N of I with

$$\mathrm{span}\,\{\varphi_i : i \in M\} \cap \mathrm{span}\,\{\varphi_i : i \in N\} = \{0\},$$

the angle between $\mathrm{span}\{\varphi_i : i \in M\}$ *and* $\mathrm{span}\{\varphi_i : i \in N\}$ *is greater than or equal to a.*

Thus we have a geometric interpretation of condition (a) in Theorems 2.3 and 2.4. In the last section, we shall also obtain an equivalent condition to (b') of Remark 2.6, which is geometrically more appealing.

Let us end this section with the observation that the conclusion in Corollary 2.5 holds under weaker assumptions for exponential systems $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$, where $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$. In fact, assuming only that $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Bessel sequence in $L^2(-\pi, \pi)$, the family $\{\lambda_k\}_{k\in\mathbb{Z}}$ is relatively separated (see e.g Theorem 3.1 in [7]). Thus $\{\lambda_k\}_{k\in\mathbb{Z}}$ can be split into a finite union of separated sets with *arbitrary* large separation constants. For sufficiently large separation constants, the subsequences will be Riesz sequences by Theorem 2.2 in [16]. Thus $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a finite union of Riesz sequences.

3 Decomposition of wavelets

In this section we prove results concerning decomposition of a wavelet system

$$\{\psi_{j,k}\}_{j,k\in\mathbb{Z}} = \{a^{j/2}\psi(a^jx - kb)\}_{j,k\in\mathbb{Z}},\ a > 1, b > 0$$

into linearly independent sets. In general, we will not assume that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a frame. But as starting point we prove that Corollary 2.5 can be strengthened for a wavelet frame arising from a frame multiresolution analysis:

Example 3.1 A frame multiresolution analysis $\{V_j, \phi\}$ is defined exactly like a MRA, except that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is assumed to be a frame for V_0 ; see [2], [3]. Assume the nested sequence $\{V_j\}_{j\in\mathbb{Z}}$ is ordered such that $V_{-1} \subset V_0 \subset V_1 \subset \cdots$ and let W_j be the orthogonal complement of V_j in V_{j+1}. In contrast to the MRA-case, the definition of a frame multiresolution analysis does not imply the existence of a function $\psi \in W_0$ for which $\{T_k\psi\}_{k\in\mathbb{Z}}$ is a frame for W_0 and $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ (equivalent conditions for this to happen can be found in [2]). Now we assume that the frame multiresolution analysis $\{V_j, \phi\}$ generates a frame $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. We want to prove that then $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is actually a Riesz basis for $L^2(\mathbb{R})$: By the construction in [3], for each $j \in \mathbb{Z}$, $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is a frame for W_j and $L^2(\mathbb{R}) = \oplus W_j$. If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz frame, then for each $j \in \mathbb{Z}$, $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is a Riesz frame for W_j. A frame of translates is automatically linearly independent when no translate is repeated [13], so by Lemma 2.2 it follows that $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is a Riesz sequence; using the orthogonal decomposition again, we conclude that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$.

Note that the above argument only excludes the existence of an overcomplete Riesz frame: there actually exist frames of the type $\{f(\cdot - k)\}_{k\in\mathbb{Z}}$ which are not Riesz bases, cf. [1]. However, by Lemma 2.2 such a frame can not be a Riesz frame. It is also known that $\{f(\cdot - k)\}_{k\in\mathbb{Z}}$ can only be a frame for a pure subspace of $L^2(\mathbb{R})$, cf. [7]. For more information about Riesz bases and linear independence we refer to [14].

Proposition 8.2 in [12] implies that if $\psi \in L^2(\mathbb{R})$ satisfies certain admissibility conditions, then $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a frame which can be decomposed into a finite union of Riesz sequences. It is an interesting open problem whether it is possible at all to find a wavelet frame, which does not have this property.

We can prove that at least not every *wavelet system* can be decomposed into a finite union of Riesz sequences. Consider for example the scaling function ϕ associated with a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$. Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is ordered increasingly. For each $j \in \mathbb{Z}$ the family $\{2^{j/2}\phi(2^j x - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j, so from $V_{-1} \subset V_0 \subset V_1 \subset \cdots$ it clearly follows that the wavelet system $\{2^{j/2}\phi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ cannot be decomposed into a finite union of Riesz sequences. Another argument is that $\{2^{j/2}\phi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ is not a Bessel sequence, which clearly excludes the "decomposition property".

The next-best will be to ask for linear independence. One can prove that for functions $\psi \in L^2(\mathbb{R})$ with continuous and compactly supported Fourier transform, the wavelet system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is linearly independent, cf. [8]. But in general $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ might be linearly dependent; with $\psi := 1_{[0,1]}$ and $a = 2, b = 1$ it is well known that $\psi_{1,0} = \frac{1}{\sqrt{2}}(\psi_{2,0} + \psi_{2,1})$. Below we give conditions implying that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ can be decomposed into a finite union of linearly independent sets:

Theorem 3.2 *Let $a > 1$, $b > 0$ and assume that $\psi \in L^2(\mathbb{R})$ has compact support. Define*

$$c := \sup \operatorname{supp} \psi - \inf \operatorname{supp} \psi,$$

and suppose there is an interval of positive length d on which $\psi \neq 0$ a.e. Let m and n be integers such that

$$m \geq \frac{2c}{b} \quad \text{and} \quad n \geq \log_a \frac{4c}{d}. \tag{3}$$

Then the wavelet system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ with parameters a and b can be decomposed into mn linearly independent sets. More precisely, for any $r \in \{1, \ldots, m\}$ and $s \in \{1, \ldots, n\}$, the set $\{\psi_{nj+s,mk+r} : j, k \in \mathbb{Z}\}$ is linearly independent.

Proof: For $j, k \in \mathbb{Z}$ set

$$p_{j,k} := \inf \operatorname{supp} \psi_{j,k}, \quad q_{j,k} := \sup \operatorname{supp} \psi_{j,k}.$$

Since $\operatorname{supp} \psi_{j,k}$ is the image of $\operatorname{supp} \psi$ under the map

$$F_{j,k} : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{x + bk}{a^j}, \tag{4}$$

we have

$$q_{j,k} - p_{j,k} = \frac{c}{a^j}.$$

Furthermore, with m chosen as in (3),

$$p_{j,k+m} - q_{j,k} = \frac{p_{0,0} - q_{0,0} + mb}{a^j} \geq \frac{c}{a^j}.$$

Thus, supp $\psi_{j,k}$ is contained in an interval of length at most $\frac{c}{a^j}$ and there is a gap of at least $\frac{c}{a^j}$ between supp $\psi_{j,k}$ and supp $\psi_{j,k+m}$.

Now fix $r \in \{1, \ldots, m\}$ and $s \in \{1, \ldots, n\}$. To show that $\{\psi_{nj+s,mk+r} : j,k \in \mathbb{Z}\}$ is linearly independent it suffices to show that for any $j_0, k_0 \in \mathbb{Z}$ there is no finite subset S of $(\{j_0, j_0+1, \ldots\} \times \mathbb{Z}) \setminus \{(j_0, k_0)\}$ and coefficients $\{\alpha_{j,k}\}_{(j,k)\in S}$ such that

$$\psi_{nj_0+s,mk_0+r} = \sum_{(j,k)\in S} \alpha_{j,k}\, \psi_{nj+s,mk+r}. \tag{5}$$

But this follows if we show that for any $j_0, j_1, k_0 \in \mathbb{Z}$ with $j_1 \geq j_0$ there is a subinterval $L_{j_1} = L_{j_1}(j_0, k_0)$ of supp ψ_{nj_0+s,mk_0+r} of positive length on which $\psi_{nj_0+s,mk_0+r} \neq 0$ and such that

$$L_{j_1} \cap \operatorname{supp} \psi_{nj+s,mk+r} = \emptyset \;\; \forall\, (j,k) \in (\{j_0, \ldots, j_1\} \times \mathbb{Z}) \setminus \{(j_0, k_0)\}.$$

In this case, (5) is clearly impossible.

For fixed $j_0, k_0 \in \mathbb{Z}$, we will do the construction of L_{j_1} by induction on $j_1 \geq j_0$ by showing that L_{j_1} can be chosen to have length at least $\frac{4c}{a^{n(j_1+1)+s}}$. For j_0 define $L_{j_0} := F_{nj_0+s,mk_0+r}(L)$, where $F_{j,k}$ is defined in (4). Then it is clear that L_{j_0} has the desired property and by (3) has length $\frac{d}{a^{nj_0+s}} \geq \frac{4c}{a^{n(j_0+1)+s}}$.

Now suppose that L_{j_1-1} has already been constructed for $j_1 \geq j_0 + 1$. If supp $\psi_{nj_1+s,mk+r} \cap L_{j_1-1} = \emptyset \;\forall\; k \in \mathbb{Z}$, then we can choose $L_{j_1} = L_{j_1-1}$. If this is not the case, there is $k_1 \in \mathbb{Z}$ such that supp $\psi_{nj_1+s,mk_1+r} \cap L_{j_1-1} \neq \emptyset$. But as we have seen, supp ψ_{nj_1+s,mk_1+r} has length $\frac{c}{a^{nj_1+s}}$, which is at most one quarter of the length of L_{j_1-1}. Also, the gap between supp ψ_{nj_1+s,mk_1+r} and supp $\psi_{nj_1+s,mk+r}$ is at least $\frac{c}{a^{nj_1+s}}$ for $k \neq k_1$. Thus there is a subinterval L_{j_1} of L_{j_1-1} of length $\frac{c}{a^{nj_1+s}} \geq \frac{4c}{a^{n(j_1+1)+s}}$ such that $L_{j_1} \cap \operatorname{supp} \psi_{nj_1+s,mk+r} = \emptyset \;\forall\; k \in \mathbb{Z}$. Clearly, L_{j_1} has the desired properties, concluding the induction step. $\square$

Theorem 3.2 proves the possibility of decomposing $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ into a finite union of linearly independent sets without assuming that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a frame; in this sense, it is more general than frame theory. However, to connect with Section 2 we note that there actually exist (overcomplete) wavelet frames $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ satisfying the conditions in Theorem 3.2. For example, take any piecewise continuous function ψ with compact support, for which $\{2^{j/2}\psi(2^jx-k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$; ψ could for example be any of the Daubechies wavelets with compact support (cf. Section 6.4 in [11]), or simply the function $1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},1[}$. Then the Oversampling Theorem by Chui and Shi [10] shows that for all odd n, $\{2^{j/2}\psi(2^jx-k/n)\}_{j,k\in\mathbb{Z}}$ is a tight wavelet frame for $L^2(\mathbb{R})$; it is clearly overcomplete because it contains the orthonormal basis $\{2^{j/2}\psi(2^jx-k)\}_{j,k\in\mathbb{Z}}$ as a proper subset.

Note that *if* $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz frame, then we do not need Theorem 3.2, because an even stronger result (namely Corollary 2.5) holds in this case. However, we believe that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is only a Riesz frame if it is already a Riesz basis:

Conjecture: If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz frame, then $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis.

The conjecture is supported by the special case discussed in Example 3.1 and by the fact that if $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz frame, then $\{a^{j/2}\psi(a^jx-kb)\}_{k\in\mathbb{Z}}$ is a Riesz frame for its closed span for all $j\in\mathbb{Z}$, and therefore a Riesz sequence by Lemma 2.2.

We note that for a Gabor frame, i.e., a frame for $L^2(\mathbb{R})$ of the form

$$\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}, \text{ where } g\in L^2(\mathbb{R}), a,b>0,$$

the analogue of the conjecture is true. This follows from the Proposition below (we state it in detail because of a print mistake in [6]) combined with the observation that if $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ is a frame, then $ab\le 1$, with equality if and only if $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ is a Riesz basis.

Proposition 3.3 *Suppose that $ab<1$ and that $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. For $N\in\mathbb{N}$, let A_N denote the optimal lower bound for $\{e^{2\pi imbx}g(x-na)\}_{|m|,|n|\le N}$. Then*

$$A_N\to 0 \text{ as } N\to\infty.$$

4 A criterion for decomposition

In this section we prove the following criterion giving the number of linearly independent subsets into which a given finite set can be decomposed. We make the convention that also the empty set $\{\varphi_j : j \in \emptyset\}$ will be regarded as a linearly independent set.

Theorem 4.1 *Let $E \in \mathbb{N}$, I be a finite set and $\{\varphi_i\}_{i\in I}$ be a sequence of non-zero elements in a vector space. Then I can be partitioned into E disjoint sets $D_1, \ldots, D_E$ for which each of the $\{\varphi_i\}_{i\in D_j}$ $(j = 1, \ldots, E)$ is linearly independent if and only if for any nonempty subset J of I we have*

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq E. \tag{6}$$

Proof: For the "only if"-part, suppose there is a partition $I = \bigcup_{j=1}^{E} D_j$ into disjoint sets such that each $\{\varphi_i\}_{i\in D_j}$ is linearly independent. Let J be any nonempty subset of I and put $J_j := J \cap D_j$ for $j = 1, \ldots, E$. Since $\{\varphi_i : i \in J_j\}$ is linearly independent for any j, we have

$$\dim \operatorname{span}\{\varphi_i : i \in J_j\} = |J_j|.$$

Thus we obtain

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_i : i \in J\}} \leq \frac{|J|}{\max_{1\leq j\leq E} \dim \operatorname{span}\{\varphi_i : i \in J_j\}} =$$

$$\frac{|J|}{\max_{1\leq j\leq E}\{|J_j|\}} = \frac{|J_1| + \ldots + |J_E|}{\max_{1\leq j\leq E}\{|J_j|\}} \leq \frac{E \cdot \max_{1\leq j\leq E}\{|J_j|\}}{\max_{1\leq j\leq E}\{|J_j|\}} = E.$$

The "if"-part will follow if we show the following stronger result:

Claim: If $\{\varphi_i\}_{i\in I}$ is a finite sequence of non-zero vectors for which (6) holds, and if $C_1, \ldots, C_E$ is a partition of I into disjoint sets, then there is a partition $D_1, \ldots, D_E$ of I into disjoint sets such that each of the sets $\{\varphi_i\}_{i\in D_j}$ is linearly independent $(j = 1, \ldots, E)$, and such that

$$G_j := \operatorname{span}\{\varphi_i : i \in C_j\} \subset \operatorname{span}\{\varphi_i : i \in D_j\} =: F_j \quad \forall\, j = 1, \ldots, E.$$

Put $G := \operatorname{span}\{\varphi_i : i \in I\}$. The proof of the claim above will be done by induction on $\dim G$, the dimension of G. The claim is trivially true for $\dim G = 0$ or $\dim G = 1$. So suppose we have $\dim G \geq 2$ and that the claim is true for all finite sequences of vectors fulfilling (6) whose linear span has dimension less than $\dim G$.

We split the proof into two cases. First, if none of the G_j's is a proper subset of G, we must have $|C_j| \geq \dim G$ for $j = 1, \dots, E$, hence $|I| = \sum_{j=1}^{E} |C_j| \geq E \cdot \dim G$. But from (6) we conclude $|I| \leq E \cdot \dim G$. Thus we have $|C_j| = \dim G = \dim G_j$ for all j, meaning that $\{\varphi_i\}_{i\in C_j}$ is already linearly independent for all j, and we can choose $D_j = C_j$ in this case.

The second case is that there is some $p \in \{1, \dots, E\}$ such that $G_p \subsetneq G$. If all $\{\varphi_i\}_{i\in C_j}$ are linearly independent, there is nothing to prove. So suppose there is $q \in \{1, \dots, E\}$ such that $\{\varphi_i\}_{i\in C_q}$ is linearly dependent. If it is possible to choose q and p to be equal, do so. If not, we have $G_q = G$. Then take some $i_q \in C_q$ such that we still have $\operatorname{span}\{\varphi_i : i \in C_q \setminus \{i_q\}\} = G_q = G$, and define

$$C'_j := \begin{cases} C_q \setminus \{i_q\} & \text{for } j = q \\ C_p \cup \{i_q\} & \text{for } j = p \\ C_j & \text{for } j \neq p, q. \end{cases}$$

Then

$$\operatorname{span}\{\varphi_i : i \in C_j\} \subset \operatorname{span}\{\varphi_i : i \in C'_j\}$$

for all j. Now we can rename G_j to be $\operatorname{span}\{\varphi_i : i \in C'_j\}$ and C'_j to be C_j. Repeating this proceedure as long as necessary, after a finite number of iterations we arrive at some situation where all $\{\varphi_i\}_{i\in C_j}$ are linearly independent (in which case we are done), or where there is some $k \in \{1, \dots, E\}$ such that $\{\varphi_i\}_{i\in C_k}$ is linearly dependent and $G_k \subsetneq G$. In the latter case define

$$S_j := \{i \in C_j : \varphi_i \in G_k\}$$

for $j = 1, \dots, E$. Then $S_1, \dots, S_E$ is a partition of $\bigcup_{j=1}^{E} S_j = \{i \in I : \varphi_i \in G_k\}$. Since

$$\dim \operatorname{span}\{\varphi_i : i \in \bigcup_{j=1}^{E} S_j\} = \dim G_k < \dim G,$$

the induction hypothesis gives a partition $S'_1, \dots, S'_E$ of $\bigcup_{j=1}^{E} S_j$ into disjoint subsets, such that $\{\varphi_i\}_{i\in S'_j}$ is linearly independent for all $j \in \{1, \dots, E\}$ and such that

$$\operatorname{span}\{\varphi_i : i \in S_j\} \subset \operatorname{span}\{\varphi_i : i \in S'_j\} \subset G_k \;\; \forall\, j \in \{1, \ldots, E\}.$$

Define for $j \in \{1, \ldots, E\}$,

$$\begin{aligned} C'_j &:= (C_j \setminus S_j) \cup S'_j, \\ G'_j &:= \operatorname{span}\{\varphi_i : i \in C'_j\}. \end{aligned}$$

Then $G_j \subset G'_j$ for all $j = 1, \ldots, E$. Furthermore, $G_k = G'_k$ and $C'_k = S'_k$. Now, if all $\{\varphi_i\}_{i \in C'_j}$ are linearly independent, we are done. If not, there must be some $m \in \{1, \ldots, E\} \setminus \{k\}$ such that $\{\varphi_i\}_{i \in C'_m}$ is linearly dependent. Since $\{\varphi_i\}_{i \in S'_m}$ is linearly independent, there must be $i_m \in C'_m \setminus S'_m$ such that

$$\operatorname{span}\{\varphi_i : i \in C'_m \setminus \{i_m\}\} = \operatorname{span}\{\varphi_i : i \in C'_m\}.$$

But $\varphi_{i_m} \notin G_k$, so $\{\varphi_i\}_{i \in C'_k \cup \{i_m\}}$ is linearly independent and its span is strictly greater than G_k. Defining

$$D_j := \begin{cases} C'_k \cup \{i_m\} & \text{for } j = k \\ C'_m \setminus \{i_m\} & \text{for } j = m \\ C'_m & \text{for } j \neq k, m, \end{cases}$$

and $F_j := \operatorname{span}\{\varphi_i : i \in D_j\}$, we see that $G_j \subset F_j$ for all j and that for at least one j the inclusion is strict. Repeating the proceedure as long as necessary (with $C_j := D_j$ and $G_j := F_j$) we finally arrive, after a finite number of steps, at the situation where all $\{\varphi_i\}_{i \in D_j}$ are linearly independent. This shows the claim. □

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Ole Christensen
Technical University of Denmark
Department of Mathematics
Building 303
2800 Lyngby, Denmark
email: Ole.Christensen@mat.dtu.dk

Alexander M. Lindner
Zentrum Mathematik
Technische Universität München
D-80290 München, Germany
email: lindner@mathematik.tu-muenchen.de