PROPERTIES OF STATIONARY DISTRIBUTIONS OF A SEQUENCE OF GENERALIZED ORNSTEIN–UHLENBECK PROCESSES

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Abstract. The infinite (in both directions) sequence of the distributions $\mu^{(k)}$ of the stochastic integrals $\int_0^{\infty-} c^{-N_{t-}^{(k)}} dL_t^{(k)}$ for integers k is investigated. Here $c > 1$ and $(N_t^{(k)}, L_t^{(k)})$, $t \geq 0$, is a bivariate compound Poisson process with Lévy measure concentrated on three points $(1,0)$, $(0,1)$, $(1, c^{-k})$. The amounts of the normalized Lévy measure at these points are denoted by p, q, r. For $k = 0$ the process $(N_t^{(0)}, L_t^{(0)})$ is marginally Poisson and $\mu^{(0)}$ has been studied by Lindner and Sato (Ann. Probab. **37** (2009), 250–274). The distributions $\mu^{(k)}$ are the stationary distributions of a sequence of generalized Ornstein–Uhlenbeck processes structurally related in some way. Continuity properties of $\mu^{(k)}$ are shown to be the same as those of $\mu^{(0)}$. The dependence on k of infinite divisibility of $\mu^{(k)}$ is clarified. The problem to find necessary and sufficient conditions in terms of c, p, q, and r for $\mu^{(k)}$ to be infinitely divisible is somewhat involved, but completely solved for every integer k. The conditions depend on arithmetical properties of c. The symmetrizations of $\mu^{(k)}$ are also studied. The distributions $\mu^{(k)}$ and their symmetrizations are c^{-1}-decomposable, and it is shown that, for each $k \neq 0$, $\mu^{(k)}$ and its symmetrization may be infinitely divisible without the corresponding factor in the c^{-1}-decomposability relation being infinitely divisible. This phenomenon was first observed by Niedbalska-Rajba (Colloq. Math. **44** (1981), 347–358) in an artificial example. The notion of quasi-infinite divisibility is introduced and utilized, and it is shown that a quasi-infinitely divisible distribution on $[0,\infty)$ can have its quasi-Lévy measure concentrated on $(-\infty, 0)$.

1. Introduction

Let $\{V_t, t \geq 0\}$ be a generalized Ornstein–Uhlenbeck process associated with a bivariate Lévy process $\{(\xi_t, \eta_t), t \geq 0\}$ with initial condition S. That is, $\{V_t\}$ is a stochastic process defined by

$$V_t = e^{-\xi_t} \left(S + \int_0^t e^{\xi_{s-}} \, d\eta_s \right), \tag{1.1}$$

where $\{(\xi_t, \eta_t)\}$ and S are assumed to be independent (Carmona et al. [4, 5]). Define two other bivariate Lévy process $\{(\xi_t, L_t)\}, t \geq 0\}$ and $\{(U_t, L_t), t \geq 0\}$ by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} = \begin{pmatrix} \xi_t - \sum_{0<s\leq t} \left(e^{-(\xi_s - \xi_{s-})} - 1 + (\xi_s - \xi_{s-}) \right) - t\, 2^{-1} \alpha_{\xi,\xi} \\ \eta_t + \sum_{0<s\leq t} (e^{-(\xi_s - \xi_{s-})} - 1)(\eta_s - \eta_{s-}) - t\, \alpha_{\xi,\eta} \end{pmatrix} \tag{1.2}$$

where $\alpha_{\xi,\xi}$ and $\alpha_{\xi,\eta}$ are the $(1,1)$ and the $(1,2)$ element of the Gaussian covariance matrix of $\{(\xi_t,\eta_t)\}$, respectively. Then $\{V_t, t\geq 0\}$ is the unique solution of the stochastic differential equation

$$dV_t = -V_{t-}\,dU_t + dL_t, \quad t\geq 0, \quad V_0 = S, \tag{1.3}$$

the filtration being such that $\{V_t\}$ is adapted and $\{U_t\}$ and $\{L_t\}$ are both semimartingales with respect to it (see Maller et. al [15], p. 428, or Protter [18], Exercise V.27). Hence we shall also refer to a generalized Ornstein–Uhlenbeck process associated with $\{(\xi_t,\eta_t)\}$ as the solution of the SDE (1.3) *driven by* $\{(U_t,L_t)\}$. Let

$$\mu = \mathcal{L}\left(\int_0^{\infty-} e^{-\xi_{s-}}\,dL_s\right), \tag{1.4}$$

whenever the improper integral exists, where $\mathcal{L}$ stands for "distribution of". If $\{\xi_t\}$ drifts to $+\infty$ as $t\to\infty$ (or, alternatively, under a minor non-degeneracy condition), a necessary and sufficient condition for $\{V_t\}$ to be a strictly stationary process under an appropriate choice of S is the almost sure convergence of the improper integral in (1.4); in this case μ is the unique stationary marginal distribution (Lindner and Maller [11]). The condition for the convergence of the improper integral in (1.4) in terms of the Lévy–Khintchine triplet of $\{(\xi_t,L_t)\}$ is given by Erickson and Maller [7]. Properties of the distribution μ are largely unknown, apart from some special cases. For example, it is selfdecomposable if $\eta_t = t$ and $\xi_t = (\log c)N_t$ for a Poisson process $\{N_t\}$ and a constant $c>1$ (Bertoin et al. [1]), or if $\{\xi_t\}$ is spectrally negative and drifts to $+\infty$ as $t\to\infty$ (see Bertoin et al. [2] and Kondo et al. [10] for a multivariate generalization). Bertoin et al. [2] have shown that the distribution in (1.4) is always continuous unless degenerated to a Dirac measure. In Lindner and Sato [12], the distribution μ in (1.4) and its symmetrization was studied for the case when $\{(\xi_t,L_t)\} = \{((\log c)N_t, L_t)\}$ for a constant $c>1$ and a bivariate Lévy process $\{(N_t,L_t)\}$ such that both $\{N_t\}$ and $\{L_t\}$ are Poisson process; the Lévy measure of $\{(N_t,L_t)\}$ is then concentrated on the three points $(1,0)$, $(0,1)$ and $(1,1)$.

In this paper we extend the setup of our paper [12], by defining a sequence of bivariate Lévy processes $\{(N_t^{(k)},L_t^{(k)}), t\geq 0\}$, $k\in\mathbb{Z}=\{\ldots,-1,0,1,\ldots\}$, in the following way. The process $\{(N_t^{(k)},L_t^{(k)})\}$ has the characteristic function

$$E[e^{i(z_1N_t^{(k)}+z_2L_t^{(k)})}] = \exp\left[t\int_{\mathbb{R}^2}(e^{i(z_1x_1+z_2x_2)}-1)\nu^{(k)}(dx)\right], \quad (z_1,z_2)\in\mathbb{R}^2, \tag{1.5}$$

where the Lévy measure $\nu^{(k)}$ is concentrated on at most three points $(1,0)$, $(0,1)$, $(1,c^{-k})$ with $c>1$ and

$$u=\nu^{(k)}(\{(1,0)\}),\quad v=\nu^{(k)}(\{(0,1)\}),\quad w=\nu^{(k)}(\{(1,c^{-k})\}).$$

We assume that $u+w>0$ and $v+w>0$, so that $\{N_t^{(k)}\}$ is a Poisson process with parameter $u+w$ and $\{L_t^{(k)}\}$ is a compound Poisson process with Lévy measure concentrated on at most two points 1, c^{-k} with total mass $v+w$. In particular, $\{L_t^{(0)}\}$ is a Poisson process with parameter $v+w$. We define the normalized Lévy measure, which has mass

$$p=\frac{u}{u+v+w},\quad q=\frac{v}{u+v+w},\quad r=\frac{w}{u+v+w}$$

at the three points. We have $p,q,r\geq 0$ and $p+q+r=1$. The assumption that $u+w>0$ and $v+w>0$ is now written as $p+r>0$ and $q+r>0$. We are interested in continuity properties and conditions for infinite divisibility of the distribution

$$\mu^{(k)}=\mathcal{L}\left(\int_0^{\infty-}c^{-N_{s-}^{(k)}}dL_s^{(k)}\right),\quad k\in\mathbb{Z}. \tag{1.6}$$

For $k=0$ the distribution $\mu^{(0)}$ is identical with the distribution $\mu_{c,q,r}$ studied in our paper [12]. As will be shown in Proposition 2.1 below, $\mu^{(k+1)}$ is the unique stationary distribution of the generalized Ornstein-Uhlenbeck process associated with $\{((\log c)N_t^{(k)},L_t^{(k)})\}$ as defined in (1.1), while $\mu^{(k)}$ appears naturally as the unique stationary distribution of the SDE (1.3) driven by $\{((1-c^{-1})N_t^{(k)},L_t^{(k)})\}$. The fact that $\{(N_t^{(k)},L_t^{(k)})\}$ is related to both $\mu^{(k+1)}$ and $\mu^{(k)}$ in a natural way explains the initial interest in the distributions $\mu^{(k)}$ with general $k\in\mathbb{Z}$. As discussed below, they have some surprising properties which cannot be observed for $k=0$.

In contrast to the situation in our paper [12], where $\mu^{(0)}$ was studied, continuity properties of $\mu^{(k)}$ are easy to handle, in the sense that they are reduced to those of $\mu^{(0)}$; but classification of $\mu^{(k)}$ into infinitely divisible and non-infinitely divisible cases is more complicated than that of $\mu^{(0)}$. We will give a complete answer to this problem. The criterion for infinite divisibility of $\mu^{(k)}$ depends on arithmetical properties of c. It is more involved for $k<0$ than for $k>0$. If $k<0$ and c^j is an integer for some positive integer j, we will have to introduce a new class of functions $h_{\alpha,\gamma}(x)$ with integer parameters $\alpha\geq 2$ and $\gamma\geq 1$ to express the criterion. In the case that $k<0$ and c^j is not an integer for any positive integer j, the hardest situation is where $c^j=3/2$ for some integer j. In this situation, however, we will express the criterion by 149 explicit inequalities between p, q, and r. It will be also shown that, for p, q, and r fixed, the infinite divisibility of $\mu^{(k)}$, $k \in \mathbb{Z}$, has the following monotonicity: if $\mu^{(k)}$ is infinitely divisible for some $k = k_0$, then $\mu^{(k)}$ is infinitely divisible for all $k \geq k_0$. Further, if $p > 0$ and $r > 0$, then $\mu^{(k)}$ is non-infinitely divisible for all k sufficiently close to $-\infty$. The case where $\mu^{(k)}$ is non-infinitely divisible for all $k \in \mathbb{Z}$ is also characterized in terms of the parameters.

The investigation of the law $\mu^{(k)}$ is related to the study of c^{-1}-decomposable distributions. For $b \in (0, 1)$ a distribution σ on $\mathbb{R}$ is said to be b-decomposable if there is a distribution ρ such that

$$\widehat{\sigma}(z) = \widehat{\rho}(z)\,\widehat{\sigma}(bz), \quad z \in \mathbb{R}.$$

Here $\widehat{\sigma}(z)$ and $\widehat{\rho}(z)$ denote the characteristic functions of σ and ρ. The "factor" ρ is not necessarily uniquely determined by σ and b, but it is if $\widehat{\sigma}(z) \neq 0$ for z from a dense subset of $\mathbb{R}$. If ρ is infinitely divisible, then so is σ, but the converse is not necessarily true as pointed out by Niedbalska-Rajba [16] in a somewhat artificial example. The study of b-decomposable distributions is made by Loève [14], Grincevičjus [9], Wolfe [21], Bunge [3], Watanabe [20], and others. In particular, any b-decomposable distribution which is not a Dirac measure is either continuous-singular or absolutely continuous ([9] or [21]).

We will show that, for $k \in \mathbb{Z}$, $\mu^{(k)}$ is c^{-1}-decomposable and explicitly give the distribution $\rho^{(k)}$ satisfying

$$\widehat{\mu}^{(k)}(z) = \widehat{\rho}^{(k)}(z)\,\widehat{\mu}^{(k)}(c^{-1}z), \tag{1.7}$$

where $\widehat{\mu}^{(k)}(z)$ and $\widehat{\rho}^{(k)}(z)$ are the characteristic functions of $\mu^{(k)}$ and $\rho^{(k)}$. The distribution $\rho^{(k)}$ is unique here as will follow from Proposition 2.3 below. A criterion for infinite divisibility of $\rho^{(k)}$ for $k \in \mathbb{Z}$ in terms of c, p, q, and r will be given; it is simpler than that of $\mu^{(k)}$. In particular, it will be shown that for every $k \neq 0$ there are parameters c, p, q, r such that the factor $\rho^{(k)}$ is not infinitely divisible while $\mu^{(k)}$ is infinitely divisible. This is different from the situation $k = 0$ treated in [12], since such a phenomenon does not happen for $\mu^{(0)}$. Allowing $k \neq 0$, we obtain a lot of examples satisfying this phenomenon, and unlike in Niedbalska-Rajba [16], our examples are connected with simple stochastic processes.

We also consider the symmetrizations $\mu^{(k)\,\mathrm{sym}}$ for general $k \in \mathbb{Z}$. Then $\mu^{(k)\,\mathrm{sym}}$ is again c^{-1}-decomposable and satisfies

$$\widehat{\mu}^{(k)\,\mathrm{sym}}(z) = \widehat{\rho}^{(k)\,\mathrm{sym}}(z)\,\widehat{\mu}^{(k)\,\mathrm{sym}}(c^{-1}z). \tag{1.8}$$

Necessary and sufficient conditions for infinite divisibility of $\mu^{(k)\,\mathrm{sym}}$ and of $\rho^{(k)\,\mathrm{sym}}$ are obtained. In particular, it will be shown that if $k \neq 0$, then $\mu^{(k)\,\mathrm{sym}}$ can be infinitely divisible without $\rho^{(k)\,\mathrm{sym}}$ being infinitely divisible, a phenomenon which does not occur for $\mu^{(0)}$ treated in [12]. The argument we use to characterize infinite divisibility of $\mu^{(k)\,\mathrm{sym}}$ for $k \in \mathbb{Z}$ is new also in the situation $k = 0$, and simplifies the proof given in [12] for that situation considerably.

We introduce the following notion for distributions having Lévy–Khintchine-like representation. A distribution σ on $\mathbb{R}$ is called *quasi-infinitely divisible* if

$$\widehat{\sigma}(z) = \exp\left[i\gamma z - az^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx 1_{[-1,1]}(x))\, \nu_\sigma(dx)\right], \tag{1.9}$$

where $\gamma, a \in \mathbb{R}$ and ν_σ is a signed measure on $\mathbb{R}$ with total variation measure $|\nu_\sigma|$ satisfying $\nu_\sigma(\{0\}) = 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1)\, |\nu_\sigma|(dx) < \infty$. The signed measure ν_σ will be called *quasi-Lévy measure* of σ. Note that γ, a and ν_σ in (1.9) are unique if they exist. Infinitely divisible distributions on $\mathbb{R}$ are quasi-infinitely divisible. A quasi-infinitely divisible distribution σ on $\mathbb{R}$ is infinitely divisible if and only if $a \geq 0$ and the negative part of ν_σ in the Jordan decomposition is zero. See E12.2 and E12.3 of [19]. We shall see in Corollary 4.2 that some of the distributions $\mu^{(k)}$, supported on $\mathbb{R}_+ = [0, \infty)$, are quasi-infinitely divisible with non-trivial quasi-Lévy measure being concentrated on $(-\infty, 0)$. Such a phenomenon does not occur in infinitely divisible case.

In this paper ID, ID^0, and ID^{00} respectively denote the class of infinitely divisible distributions on $\mathbb{R}$, the class of quasi-infinitely divisible, non-infinitely divisible distributions on $\mathbb{R}$, and the class of distributions on $\mathbb{R}$ which are not quasi-infinitely divisible. When characterizing infinite divisibility of $\rho^{(k)}$, $\mu^{(k)}$, $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$ we shall more precisely determine to which of the classes ID, ID^0 and ID^{00} the corresponding distributions belong.

Without the name of quasi-infinitely divisible distributions, the property that σ satisfies (1.9) with ν_σ having non-trivial negative part is known to be useful in showing that σ is not infinitely divisible, in books and papers such as Gnedenko and Kolmogorov [8] (p. 81), Linnik and Ostrovskii [13] (Chap. 6, § 7) and Niedbalska-Rajba [16]. We single out the class ID^0 for two reasons. The first is that μ in ID^0 has a manageable characteristic function, which is the quotient of two infinitely divisible characteristic functions. The second is that the notion is useful in studying the symmetrization μ^{sym} of μ. Already in Gnedenko and Kolmogorov [8] p. 82 an example of $\mu \notin ID$ satisfying $\mu^{\mathrm{sym}} \in ID$ is given in this way. It is noticed in [12] that

$\mu^{(0)\,\mathrm{sym}}$ (or $\rho^{(0)\,\mathrm{sym}}$) can be in ID without $\mu^{(0)}$ (or $\rho^{(0)}$) being in ID. We will show the same phenomenon occurs also for $\mu^{(k)}$ and $\rho^{(k)}$.

The paper is organized as follows: in Section 2 we describe the c^{-1}-decomposability of $\mu^{(k)}$, $k \in \mathbb{Z}$, and its consequences. Section 3 deals with continuity properties of $\mu^{(k)}$, $k \in \mathbb{Z}$. In Sections 4, 5, and 6 results on infinite divisibility and quasi-infinite divisibility of $\rho^{(k)}$ and $\mu^{(k)}$ are given for general k, positive k, and negative k, respectively. The last Section 7 discusses the symmetrizations.

We shall assume throughout the paper that $c > 1$, $p+r > 0$ and $q+r > 0$ without further mentioning. The following notation will be used. $\mathbb{N}$ (resp. $\mathbb{N}_0$) is the set of positive (resp. nonnegative) integers. $\mathbb{N}_{\mathrm{even}}$ (resp. $\mathbb{N}_{\mathrm{odd}}$) is the set of even (resp. odd) positive integers. The Lebesgue measure of B is denoted by $\mathrm{Leb}\,(B)$. The dimension of a measure σ, written $\dim(\sigma)$, is the infimum of $\dim B$, the Hausdorff dimension of B, over all Borel sets B having full σ measure. $H(\rho)$ is the entropy of a discrete measure ρ. $\mathcal{B}(\mathbb{R})$ is the class of Borel sets in $\mathbb{R}$. The Dirac measure at a point x is denoted by δ_x.

2. The c^{-1}-decomposability and its consequences

We start with the following proposition which clarifies the relations between $\{(N_t^{(k)}, L_t^{(k)})\}$, $\{(N_t^{(k-1)}, L_t^{(k-1)})\}$ and $\mu^{(k)}$.

Proposition 2.1. *Let c, p, q, r be fixed and let $k \in \mathbb{Z}$. Then*

$$\{(N_t^{(k)}, L_t^{(k)})\} \stackrel{\mathrm{d}}{=} \{(N_t^{(k-1)}, L_t^{(k-1)} + \sum_{0<s\le t} (e^{-\log(c)(N_s^{(k-1)} - N_{s-}^{(k-1)})} - 1)(L_s^{(k-1)} - L_{s-}^{(k-1)}))\},$$

so that $\{((1-c^{-1})N_t^{(k)}, L_t^{(k)})\}$ is equal in distribution to the right-hand-side of (1.2) *when applied with $\{(\xi_t, \eta_t)\} = \{(\log(c)N_t^{(k-1)}, L_t^{(k-1)})\}$. The integral $\int_0^{\infty-} c^{-N_{s-}^{(k)}}\, dL_s^{(k)}$ exists as an almost sure limit, and its distribution $\mu^{(k)}$ is the unique stationary distribution of the generalized Ornstein–Uhlenbeck process associated with $\{((\log c)N_t^{(k-1)}, L_t^{(k-1)})\}$ as defined in* (1.1), *equivalently $\mu^{(k)}$ is the unique stationary distribution of the SDE* (1.3) *driven by $\{((1-c^{-1})N_t^{(k)}, L_t^{(k)})\}$.*

Proof. The process $\{(N_t^{(k-1)}, L_t^{(k-1)})\}$ is a bivariate compound Poisson process. Its jump size is determined by the normalized Lévy measure and for $k \in \mathbb{Z}$ we have

$$\begin{aligned}\{(N_t^{(k)}, L_t^{(k)})\} &\stackrel{\mathrm{d}}{=} \{(N_t^{(k-1)}, \sum_{0<s\le t} c^{-(N_s^{(k-1)} - N_{s-}^{(k-1)})}(L_s^{(k-1)} - L_{s-}^{(k-1)}))\} \\ &= \{(N_t^{(k-1)}, L_t^{(k-1)} + \sum_{0<s\le t} (c^{-(N_s^{(k-1)} - N_{s-}^{(k-1)})} - 1)(L_s^{(k-1)} - L_{s-}^{(k-1)}))\},\end{aligned}$$

giving the first relation. The existence of the improper stochastic integral follows from the law of large numbers. The remaining assertions are then clear from the discussion in the introduction, where (1.4) was identified as the unique stationary distribution of the corresponding stochastic process. □

Let T be the first jump time for $\{N_t^{(k)}\}$ and let

$$\rho^{(k)} = \mathcal{L}(L_T^{(k)}). \tag{2.1}$$

Proposition 2.2. *For $k \in \mathbb{Z}$ the distribution $\mu^{(k)}$ is c^{-1}-decomposable and satisfies (1.7). The characteristic function of $\mu^{(k)}$ has expression*

$$\widehat{\mu}^{(k)}(z) = \prod_{n=0}^{\infty} \widehat{\rho}^{(k)}(c^{-n}z), \quad z \in \mathbb{R}. \tag{2.2}$$

Proof. By the strong Markov property for Lévy processes we have

$$\begin{aligned}\int_0^{\infty-} c^{-N_{s-}^{(k)}} dL_s^{(k)} &= L_T^{(k)} + c^{-1} \int_{T+}^{\infty-} c^{-(N_{s-}^{(k)} - N_T^{(k)})} d(L_\cdot^{(k)} - L_T^{(k)})_s \\ &\overset{\mathrm{d}}{=} L_T^{(k)} + c^{-1} \int_0^{\infty-} c^{-N_{s-}^{(k)\prime}} dL_s^{(k)\prime},\end{aligned}$$

where $\{(N_t^{(k)\prime}, L_t^{(k)\prime})\}$ is an independent copy of $\{(N_t^{(k)}, L_t^{(k)})\}$. This shows (1.7) and hence $\mu^{(k)}$ is c^{-1}-decomposable. Since (1.7) implies

$$\widehat{\mu}^{(k)}(z) = \widehat{\mu}^{(k)}(c^{-l}z) \prod_{n=0}^{l-1} \widehat{\rho}^{(k)}(c^{-n}z), \qquad z \in \mathbb{R},\ l \in \mathbb{N},$$

we obtain (2.2). □

Proposition 2.3. *For $k \in \mathbb{Z}$ the distributions $\rho^{(k)}$ and $\mu^{(k)}$ satisfy the following.*

$$\rho^{(k)} = \sum_{m=0}^{\infty} q^m p\, \delta_m + \sum_{m=0}^{\infty} q^m r\, \delta_{m+c^{-k}}, \tag{2.3}$$

$$\widehat{\rho}^{(k)}(z) = \frac{p + re^{ic^{-k}z}}{1 - qe^{iz}}, \tag{2.4}$$

$$\widehat{\mu}^{(k)}(z) = \prod_{n=0}^{\infty} \frac{p + re^{ic^{-k-n}z}}{1 - qe^{ic^{-n}z}}, \tag{2.5}$$

$$\widehat{\mu}^{(k)}(z) = \widehat{\mu}^{(k+1)}(z) \left(\frac{p}{p+r} + \frac{r}{p+r} e^{ic^{-k}z} \right), \tag{2.6}$$

$$\mu^{(k)}(B) = \frac{p}{p+r}\mu^{(k+1)}(B) + \frac{r}{p+r}\mu^{(k+1)}(B - c^{-k}), \quad B \in \mathcal{B}(\mathbb{R}), \tag{2.7}$$

$$\widehat{\mu}^{(k+1)}(z) = \widehat{\mu}^{(k)}(c^{-1}z) \frac{1-q}{1-qe^{iz}}. \tag{2.8}$$

Further, the distribution $\rho^{(k)}$ is uniquely determined by $\mu^{(k)}$ and (1.7)*.*

Proof. Let $S_1, S_2, \ldots$ be the successive jump sizes of the compound Poisson process $\{(N_t^{(k)}, L_t^{(k)})\}$. Then

$$\begin{aligned}\rho^{(k)} &= P[S_1=(1,0)]\,\delta_0 + P[S_1=(1,c^{-k})]\,\delta_{c^{-k}} \\ &\quad + \sum_{m=1}^{\infty} P[S_1=(0,1),\ldots,S_m=(0,1),S_{m+1}=(1,0)]\,\delta_m \\ &\quad + \sum_{m=1}^{\infty} P[S_1=(0,1),\ldots,S_m=(0,1),S_{m+1}=(1,c^{-k})]\,\delta_{m+c^{-k}},\end{aligned}$$

which is equal to the right-hand side of (2.3). Note that $q = 1-(p+r) < 1$. It follows from (2.3) that

$$\widehat{\rho}^{(k)}(z) = \sum_{m=0}^{\infty} q^m p e^{imz} + \sum_{m=0}^{\infty} q^m r e^{i(m+c^{-k})z},$$

which is written to (2.4). This, combined with (2.2), gives (2.5). It follows from (2.5) that

$$\begin{aligned}\widehat{\mu}^{(k)}(z) &= \lim_{l\to\infty} \prod_{n=0}^{l} \frac{p+re^{ic^{-k-n}z}}{1-qe^{ic^{-n}z}} \\ &= \lim_{l\to\infty} \frac{p+re^{ic^{-k}z}}{p+re^{ic^{-k-1-l}z}} \prod_{n=0}^{l} \frac{p+re^{ic^{-k-1-n}z}}{1-qe^{ic^{-n}z}} \\ &= \frac{p+re^{ic^{-k}z}}{p+r}\widehat{\mu}^{(k+1)}(z).\end{aligned}$$

This is (2.6). It means that $\mu^{(k)}$ is a mixture of $\mu^{(k+1)}$ with the translation of $\mu^{(k+1)}$ by c^{-k}, as in (2.7). Similarly,

$$\begin{aligned}\widehat{\mu}^{(k+1)}(z) &= \lim_{l\to\infty} \prod_{n=0}^{l} \frac{p+re^{ic^{-k-1-n}z}}{1-qe^{ic^{-n}z}} \\ &= \lim_{l\to\infty} \frac{1-qe^{ic^{-l-1}z}}{1-qe^{iz}} \prod_{n=0}^{l} \frac{p+re^{ic^{-k-1-n}z}}{1-qe^{ic^{-n-1}z}} \\ &= \frac{1-q}{1-qe^{iz}}\widehat{\mu}^{(k)}(c^{-1}z),\end{aligned}$$

which is (2.8). Finally, since

$$\sum_{n=0}^{\infty} \left| \frac{p+re^{ic^{-k-n}z}}{1-qe^{ic^{-n}z}} - 1 \right| \le \sum_{n=0}^{\infty} \frac{r|e^{ic^{-k-n}z}-1| + q|e^{ic^{-n}z}-1|}{1-q} < \infty,$$

the infinite product in (2.5) cannot be zero unless $p + re^{ic^{-k-n}z} = 0$ for some $n \in \mathbb{N}_0$. It follows that $\widehat{\mu}^{(k)}(z) \neq 0$ for z from a dense subset of $\mathbb{R}$, so that $\rho^{(k)}$ is uniquely determined by $\mu^{(k)}$ and (1.7). $\square$

3. Continuity properties for all k

Continuity properties for $\mu^{(k)}$ do not depend on k, as the following theorem shows. As a consequence of Proposition 2.3, $\mu^{(k)}$ is a Dirac measure if and only if $r = 1$. If $r < 1$, then $\mu^{(k)}$ is either continuous-singular or absolutely continuous, since it is c^{-1}-decomposable.

Theorem 3.1. *Let c, p, q, r be fixed and let $k \in \mathbb{Z}$. Then:*

(i) *$\mu^{(k)}$ is absolutely continuous if and only if $\mu^{(0)}$ is absolutely continuous.*

(ii) *$\mu^{(k)}$ is continuous-singular if and only if $\mu^{(0)}$ is continuous-singular.*

(iii) $\dim(\mu^{(k)}) = \dim(\mu^{(0)})$.

Proof. It is enough to show that absolute continuity, continuous-singularity, and the dimension of $\mu^{(k)}$ do not depend on k. We use (2.7).

(i) If $p = 0$, then $\mu^{(k)}$ is a translation of $\mu^{(k+1)}$ and the assertion is obvious. Assume that $p > 0$. Let $\mu^{(k+1)}$ be absolutely continuous. If B is a Borel set with $\mathrm{Leb}(B) = 0$, then $\mu^{(k+1)}(B) = 0$, $\mathrm{Leb}(B - c^{-k}) = 0$, and $\mu^{(k+1)}(B - c^{-k}) = 0$ and hence $\mu^{(k)}(B) = 0$ from (2.7). Hence $\mu^{(k)}$ is absolutely continuous. Conversely, let $\mu^{(k)}$ be absolutely continuous. If B is a Borel set with $\mathrm{Leb}(B) = 0$, then $\mu^{(k)}(B) = 0$ and hence $\mu^{(k+1)}(B) = 0$ from (2.7) and from $p > 0$. Hence $\mu^{(k+1)}$ is absolutely continuous.

(ii) We know that $\mu^{(k)}$ is a Dirac measure if and only if $\mu^{(0)}$ is. Hence (ii) is equivalent to (i).

(iii) We may assume $p > 0$. Let $d^{(k)} = \dim(\mu^{(k)})$. For any $\varepsilon > 0$ there is a Borel set B such that $\mu^{(k+1)}(B) = 1$ and $\dim B < d^{(k+1)} + \varepsilon$. Since

$$\begin{aligned}\mu^{(k)}(B \cup (B + c^{-k})) &= \frac{p}{p+r}\mu^{(k+1)}(B \cup (B + c^{-k})) + \frac{r}{p+r}\mu^{(k+1)}((B - c^{-k}) \cup B) \\ &\geq \frac{p}{p+r}\mu^{(k+1)}(B) + \frac{r}{p+r}\mu^{(k+1)}(B) = 1,\end{aligned}$$

we have $\mu^{(k)}(B \cup (B + c^{-k})) = 1$. Since $\dim(B \cup (B + c^{-k})) = \dim B$, this shows $d^{(k)} \leq d^{(k+1)}$. On the other hand, for any $\varepsilon > 0$ there is a Borel set E such that $\mu^{(k)}(E) = 1$ and $\dim E < d^{(k)} + \varepsilon$. If $\mu^{(k+1)}(E) < 1$, then

$$\mu^{(k)}(E) < \frac{p}{p+r} + \frac{r}{p+r}\mu^{(k+1)}(E - c^{-k}) \leq 1,$$

a contradiction. Hence $\mu^{(k+1)}(E) = 1$ and $d^{(k+1)} \le d^{(k)}$. □

By virtue of Theorem 3.1, all results on continuity properties of $\mu^{(0)}$ in [12] are applicable to $\mu^{(k)}$, $k \in \mathbb{Z}$. Thus, by the method of Erdős [6], $\mu^{(k)}$ is continuous-singular if c is a Pisot–Vijayaraghavan number and $q > 0$ (see the survey [17] on this class of numbers). On the other hand, for almost all c in $(1, \infty)$, sufficient conditions for absolute continuity of $\mu^{(k)}$ are given by an essential use of results of Watanabe [20] (see [12]).

Recall that for any discrete probability measure σ the entropy $H(\sigma)$ is defined by

$$H(\sigma) = -\sum_{x \in C} \sigma(\{x\}) \log \sigma(\{x\}),$$

where C is the set of points of positive σ measure.

Theorem 3.2. *Let c, p, q, r be fixed and let $k \in \mathbb{Z}$. We have*

$$\dim(\mu^{(k)}) \le H(\rho^{(k)})/\log c \tag{3.1}$$

and

$$H(\rho^{(k)}) \le H(\rho^{(1)}). \tag{3.2}$$

More precisely,

$$H(\rho^{(k)}) \begin{cases} = H(\rho^{(1)}) & \text{if } k > 0, \\ = H(\rho^{(1)}) & \text{if } k < 0 \text{ and } c^{-k} \notin \mathbb{N}, \\ < H(\rho^{(1)}) & \text{if } k \le 0,\ c^{-k} \in \mathbb{N}, \text{ and } p, q, r > 0. \end{cases} \tag{3.3}$$

Proof. The inequality (3.1) follows from Theorem 2.2 of Watanabe [20]. If $k > 0$ or if $k \le 0$ and $c^{-k} \notin \mathbb{N}$, then, in the expression (2.3) of $\rho^{(k)}$, all k and $k + c^{-k}$ for $k \in \mathbb{N}_0$ are distinct points and hence $H(\rho^{(k)})$ does not depend on k. For general $k \in \mathbb{Z}$, some of the points k and $k + c^{-k}$ for $k \in \mathbb{N}_0$ may coincide, which makes the entropy smaller than or equal to $H(\rho^{(1)})$. This proves (3.2). If $k \le 0$, $c^{-k} \in \mathbb{N}$, and $p, q, r > 0$, then some of points with positive mass indeed amalgamate and the entropy becomes smaller than $H(\rho^{(1)})$. □

A straightforward calculus gives

$$H(\rho^{(1)}) = (-p \log p - q \log q - r \log r)/(1 - q), \tag{3.4}$$

with the interpretation $x \log x = 0$ for $x = 0$.

Theorem 3.3. *Let $r < 1$. If $\log c > (\log 3)/(1 - q)$, then $\mu^{(k)}$ is continuous-singular for all $k \in \mathbb{Z}$.*

Proof. It follows from (3.4) that $H(\rho^{(1)}) \le (\log 3)/(1-q)$. Hence by Theorem 3.2 $\dim(\mu^{(k)}) < 1$ if $\log c > (\log 3)/(1-q)$. $\square$

4. General results on infinite divisibility for all k

We give two theorems concerning the classification of $\rho^{(k)}$ and $\mu^{(k)}$, $k \in \mathbb{Z}$, into ID, ID^0, and ID^{00}. The first theorem concerns $\rho^{(k)}$ and $\mu^{(k)}$, while the second deals with $\mu^{(k)}$. We also obtain examples of quasi-infinitely divisible distributions on $\mathbb{R}_+$ with quasi-Lévy measure being concentrated on $(-\infty, 0)$.

Theorem 4.1. (i) *If $p = 0$ or if $r = 0$, then $\rho^{(k)}$ and $\mu^{(k)}$ are in ID for every $k \in \mathbb{Z}$.*

(ii) *If $0 < r < p$, then $\rho^{(k)}$ and $\mu^{(k)}$ are in $ID \cup ID^0$ for every $k \in \mathbb{Z}$, with quasi-Lévy measures being concentrated on $(0, \infty)$.*

(iii) *If $0 < p < r$, then $\rho^{(k)}$ and $\mu^{(k)}$ are in ID^0 for every $k \in \mathbb{Z}$.*

(iv) *If $p = r > 0$, then $\rho^{(k)}$ and $\mu^{(k)}$ are in ID^{00} for every $k \in \mathbb{Z}$.*

It is noteworthy that in this theorem the results do not depend on k and the results for $\rho^{(k)}$ and $\mu^{(k)}$ are the same. By virtue of this theorem, in the classification of $\rho^{(k)}$ and $\mu^{(k)}$, it remains only to find, in the case $0 < r < p$, necessary and sufficient conditions for their infinite divisibility.

Proof of Theorem 4.1. If $r = 0$, then (2.3) shows that $\rho^{(k)}$ does not depend on k, is a geometric distribution, and hence in ID, which implies that $\mu^{(k)}$ does not depend on k and is in ID. If $p = 0$, then (2.3) shows that $\rho^{(k)}$ is a shifted geometric distribution, and hence in ID, implying that $\mu^{(k)} \in ID$. Hence (i) is true.

Let us prove (ii). Assume that $0 < r < p$. It follows from (2.4) that

$$\begin{aligned}
\widehat{\rho}^{(k)}(z) &= \exp\left[-\log(1 - qe^{iz}) + \log(1 + (r/p)e^{ic^{-k}z}) + \log p\right] \\
&= \exp\left[\sum_{m=1}^{\infty}\left(m^{-1}q^m e^{imz} - m^{-1}(-r/p)^m e^{imc^{-k}z}\right) + \log p\right] \\
&= \exp\left[\sum_{m=1}^{\infty}\left(m^{-1}q^m(e^{imz} - 1) - m^{-1}(-r/p)^m(e^{imc^{-k}z} - 1)\right) + \text{const}\right]
\end{aligned}$$

and, letting $z = 0$, we see that the constant is zero. Hence

$$\widehat{\rho}^{(k)}(z) = \exp\left[\int_{(0,\infty)} (e^{izx} - 1)\nu_{\rho^{(k)}}(dx)\right], \tag{4.1}$$

where $\nu_{\rho^{(k)}}$ is a signed measure given by

$$\nu_{\rho^{(k)}} = \sum_{m=1}^{\infty} \left[m^{-1} q^m \delta_m + (-1)^{m+1} m^{-1} (r/p)^m \delta_{mc^{-k}} \right] \tag{4.2}$$

with finite total variation. Then it follows from (2.2) that

$$\widehat{\mu}^{(k)}(z) = \exp \left[\int_{(0,\infty)} (e^{izx} - 1) \nu_{\mu^{(k)}}(dx) \right] \tag{4.3}$$

with

$$\nu_{\mu^{(k)}} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[m^{-1} q^m \delta_{mc^{-n}} + (-1)^{m+1} m^{-1} (r/p)^m \delta_{mc^{-k-n}} \right]. \tag{4.4}$$

Notice that $\int_{(0,\infty)} x \left| \nu_{\mu^{(k)}} \right| (dx) < \infty$. Hence $\rho^{(k)}$ and $\mu^{(k)}$ are in $ID \cup ID^0$ for every $k \in \mathbb{Z}$, and the quasi-Lévy measures are concentrated on $(0, \infty)$ by (4.2) and (4.4).

To prove (iii), assume $0 < p < r$. Then by (2.4) and a calculation similar to the one which lead to (4.1)

$$\begin{aligned} \widehat{\rho}^{(k)}(z) &= \frac{1 + (p/r) e^{-ic^{-k}z}}{1 - q e^{iz}} r e^{ic^{-k}z} \\ &= \exp \left[-\log(1 - q e^{iz}) + \log(1 + (p/r) e^{-ic^{-k}z}) + \log r + i c^{-k} z \right] \\ &= \exp \left[\sum_{m=1}^{\infty} m^{-1} q^m e^{imz} - \sum_{m=1}^{\infty} m^{-1} (-p/r)^m e^{-imc^{-k}z} + \log r + i c^{-k} z \right]. \end{aligned}$$

Thus

$$\widehat{\rho}^{(k)}(z) = \exp \left[\int_{\mathbb{R}} (e^{izx} - 1) \nu_{\rho^{(k)}}(dx) + i c^{-k} z \right], \tag{4.5}$$

where

$$\nu_{\rho^{(k)}} = \sum_{m=1}^{\infty} m^{-1} q^m \delta_m + \sum_{m=1}^{\infty} (-1)^{m+1} m^{-1} (p/r)^m \delta_{-mc^{-k}}. \tag{4.6}$$

Clearly the negative part in the Jordan decomposition of $\nu_{\rho^{(k)}}$ is non-zero. Hence $\rho^{(k)} \in ID^0$. As in (ii), this together with (2.2) implies

$$\widehat{\mu}^{(k)}(z) = \exp \left[\int_{\mathbb{R}} (e^{izx} - 1) \nu_{\mu^{(k)}}(dx) + i \sum_{n=0}^{\infty} c^{-k-n} z \right] \tag{4.7}$$

with

$$\nu_{\mu^{(k)}} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(m^{-1} q^m \delta_{c^{-n}m} + (-1)^{m+1} m^{-1} (p/r)^m \delta_{-mc^{-k-n}} \right). \tag{4.8}$$

Again we have $\int_{\mathbb{R}} |x|\, |\nu_{\mu^{(k)}}|(dx) < \infty$. Hence $\mu^{(k)} \in ID \cup ID^0$. If $\mu^{(k)} \in ID$, then not only $\nu_{\mu^{(k)}}$ is non-negative but also $\nu_{\mu^{(k)}}$ is concentrated on $(0,\infty)$, since $\mu^{(k)}$ is concentrated on $\mathbb{R}_+$. However

$$\int_{(-\infty,0)} |x|\, \nu_{\mu^{(k)}}(dx) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} mc^{-k-n}(-1)^{m+1}m^{-1}(p/r)^m$$
$$= \sum_{n=0}^{\infty} c^{-k-n}\sum_{m=1}^{\infty}(-1)(-p/r)^m = \frac{c^{-k}}{1-c^{-1}}\frac{p/r}{1+p/r} \neq 0.$$

Thus $\nu_{\mu^{(k)}}$ is not concentrated on $(0,\infty)$. It follows that $\mu^{(k)} \notin ID$.

To show (iv), observe that if $p = r > 0$, then $\widehat{\rho}^{(k)}(z) = 0$ and $\widehat{\mu}^{(k)}(z) = 0$ for $z = c^k\pi$ from (2.4) and (1.7), which implies that $\rho^{(k)}$ and $\mu^{(k)}$ are in ID^{00}. □

The assertion (iii) is new even in the case $k = 0$. In Theorem 2.2 of [12] it was only shown that $\mu^{(0)} \notin ID$ if $0 < p < r$ by using the representation $e^{-\varphi(\theta)}$ for the Laplace transform of infinitely divisible distributions on $\mathbb{R}_+$ with $\varphi'(\theta)$ being completely monotone. The present proof of Theorem 4.1 (iii) is simpler and shows even that $\mu^{(0)} \in ID^0$.

It is worth noting that in contrast to infinitely divisible distributions, whose Lévy measure must be concentrated on $(0,\infty)$ if the distribution itself is concentrated on $\mathbb{R}_+$, the proof of the previous Theorem shows that the same conclusion does not hold for quasi-infinitely divisible distributions. Even more surprising, the quasi-Lévy measure of a quasi-infinitely divisible distribution on $\mathbb{R}_+$ can be concentrated on $(-\infty,0)$.

Corollary 4.2. *Let $q = 0$, $0 < p < r$ and $k \in \mathbb{Z}$. Then $\rho^{(k)}$ and $\mu^{(k)}$ both have bounded support contained in $\mathbb{R}_+$, but the quasi-Lévy measures $\nu_{\rho^{(k)}}$ and $\nu_{\mu^{(k)}}$ in the Lévy–Khintchine-like representations* (4.5) *and* (4.7)*, respectively, are concentrated on $(-\infty,0)$.*

Proof. Since $q = 0$, $\rho^{(k)}$ has distribution supported on two points 0, c^{-k} and $\mu^{(k)}$ is a scaled infinite Bernoulli convolution, which is supported on $[0, c^{1-k}/(c-1)]$. The assertion on the quasi-Lévy measures is immediate from (4.5) – (4.8). □

Actually, in the Gnedenko–Kolmogorov book [8], the example in p. 81 gives, after shifting by $+1$, the distribution obtained by deleting some mass at the point 0 from a geometric distribution and by normalizing. It coincides with $\rho^{(0)}$ for $0 < p < r$ and $q > 0$.

Remark. Consider $\sigma = p\delta_0 + r\delta_1$ with $0 < p < 1$ and $r = 1 - p$. If $p = r = 1/2$, then $\sigma \in ID^{00}$. If $p > r$, then $\sigma \in ID^0$ with quasi-Lévy measure supported on $\mathbb{N}$. If $p < r$, then $\sigma \in ID^0$ with quasi-Lévy measure supported on $-\mathbb{N}$. Indeed, $\sigma = \rho^{(0)}$ with $q = 0$ and the proof of Theorem 4.1 shows this fact. By scaling and shifting, we see that any distribution supported on two points in $\mathbb{R}$ has similar properties.

We have the following monotonicity property of $\mu^{(k)}$ in k.

Theorem 4.3. *Let $k \in \mathbb{Z}$ and the parameters c, p, q, r be fixed. If $\mu^{(k)} \in ID$, then $\mu^{(k+1)} \in ID$.*

Proof. Recall the relation (2.8) in Proposition 2.3. The factor $(1-q)(1-qe^{iz})^{-1}$ is the characteristic function of a geometric distribution if $q > 0$ and of δ_0 if $q = 0$, both of which are infinitely divisible. Hence the proof is straightforward. □

5. Conditions for infinite divisibility for $k > 0$

In the classification of $\rho^{(k)}$ and $\mu^{(k)}$ into ID, ID^0, and ID^{00}, it remains only to find a necessary and sufficient condition for infinite divisibility in the case $0 < r < p$ (see Theorem 4.1).

The results in [12] show the following for $\rho^{(0)}$ and $\mu^{(0)}$.

Proposition 5.1. *Assume* $0 < r < p$. (i) *If $r \le pq$, then $\rho^{(0)}$, $\mu^{(0)} \in ID$.*
(ii) *If $r > pq$, then $\rho^{(0)}$, $\mu^{(0)} \in ID^0$.*

We stress that $\rho^{(0)} \in ID$ and $\mu^{(0)} \in ID$ are equivalent and that the classification does not depend on c.

For k a nonzero integer, to find the infinite divisibility condition is harder. The condition depends on c, and $\mu^{(k)} \in ID$ does not necessarily imply $\rho^{(k)} \in ID$.

Theorem 5.2. *Let $k \in \mathbb{N}$. Assume $0 < r < p$. Then $\rho^{(k)} \in ID$ if and only if $c^k = 2$ and $r^2 \le p^2 q$.*

Proof. For later use in the proof of Theorem 6.1 allow for the moment that $k \in \mathbb{Z}$. We have shown the expression (4.1) for $\widehat{\rho}^{(k)}(z)$ with the signed measure $\nu_{\rho^{(k)}}$ of (4.2). We have $\rho^{(k)} \in ID$ if and only if $\nu_{\rho^{(k)}} \ge 0$. If $s := 2c^{-k} \notin \mathbb{N}$, then $\rho^{(k)} \in ID^0$, since $\nu_{\rho^{(k)}}(\{2c^{-k}\}) = -2^{-1}(r/p)^2 < 0$. If $2c^{-k} = s \in \mathbb{N}$, then

$$\rho^{(k)} \in ID \quad \Leftrightarrow \quad \sum_{m=1}^{\infty} \left[m^{-1} q^m \delta_m + m^{-1}(-1)^{m+1}(r/p)^m \delta_{sm/2} \right] \ge 0$$

$$\Leftrightarrow \quad (sm)^{-1}q^{sm} \geq (2m)^{-1}(r/p)^{2m}, \quad \forall\, m \in \mathbb{N},$$

which is equivalent to

$$q^s \geq (s/2)^{1/m}(r/p)^2, \quad \forall\, m \in \mathbb{N}. \tag{5.1}$$

Now assume that $k > 0$. Then necessarily $s = 1$ and (5.1) is equivalent to $q \geq (r/p)^2$. Thus the proof is complete. □

Theorem 5.3. *Let $k \in \mathbb{N}$. Assume $0 < r < p$. Then $\mu^{(k)} \in ID$ if and only if one of the following holds:* (a) *$r \leq pq$;* (b) *$c^l = 2$ for some $l \in \{1, 2, \ldots, k\}$ and $r^2 \leq p^2 q$.*

Proof. Keep in mind the assumption $0 < r < p$. If $q = 0$, then $\mu^{(k)} \in ID^0$ since it has compact support without being a Dirac measure as shown in the proof of Corollary 4.2, hence cannot be in ID. If (a) holds, then $\mu^{(k)} \in ID$ by Theorem 4.3, since $\mu^{(0)} \in ID$. If (b) holds, then $\mu^{(k)} \in ID$ by Theorem 4.3, since $\mu^{(l)} \in ID$ by Theorem 5.2. In view of these facts and Theorem 4.1, in order to prove our theorem, it is enough to show the following two facts:
(A) *If $q > 0$, $r > pq$, and $c^l \neq 2$ for all $l \in \{1, 2, \ldots, k\}$, then $\mu^{(k)} \in ID^0$.*
(B) *If $q > 0$ and $r^2 > p^2 q$, then $\mu^{(k)} \in ID^0$.*

Suppose that $q > 0$ and $r > pq$. We have the expression (4.3) of $\widehat{\mu}^{(k)}(z)$ with the signed measure $\nu_{\mu^{(k)}}$ of (4.4). Hence

$$\nu_{\mu^{(k)}} = \sum_{n=k}^{\infty} \sum_{m=1}^{\infty} a_m \delta_{c^{-n}m} + \sum_{n=0}^{k-1} \sum_{m=1}^{\infty} m^{-1} q^m \delta_{c^{-n}m}, \tag{5.2}$$

with

$$a_m = m^{-1}(q^m - (-r/p)^m). \tag{5.3}$$

Observe that

$$a_m < 0, \qquad \forall\, m \in \mathbb{N}_{\text{even}}. \tag{5.4}$$

In order to prove (A), assume further that $c^l \neq 2$ for all $l \in \{1, 2, \ldots, k\}$. If c^l is irrational for all $l \in \mathbb{N}$, then $c^{-n}m \neq c^{-n'}m'$ whenever $(m, n) \neq (m', n')$, and we see that the negative part of $\nu_{\mu^{(k)}}$ is nonzero, and hence $\mu^{(k)} \in ID^0$. So suppose that c^l is rational for some $l \in \mathbb{N}$ and let l_0 be the smallest such l. Denote $c^{l_0} = \alpha/\beta$ with $\alpha, \beta \in \mathbb{N}$ having no common divisor. As in Case 2 in the proof of Theorem 2.2 (b) of [12], let f be the largest $t \in \mathbb{N}_0$ such that 2^t divides β. Let $m \in \mathbb{N}_{\text{even}}$ and denote

$$G_m = \{(n', m') \in \mathbb{N}_0 \times \mathbb{N} \colon c^{-n'}m' = m,\ m' \text{ odd}\},$$
$$H_m = \{(n', m') \in \mathbb{N}_0 \times \mathbb{N} \colon c^{-n'}m' = m,\ m' \text{ even}\},$$

$$G_m^{(k)} = \{(n', m') \in \{k, k+1, \dots\} \times \mathbb{N} \colon c^{-n'} m' = c^{-k} m,\ m' \text{ odd}\},$$
$$H_m^{(k)} = \{(n', m') \in \{k, k+1, \dots\} \times \mathbb{N} \colon c^{-n'} m' = c^{-k} m,\ m' \text{ even}\}.$$

Then $(n', m') \in G_m^{(k)}$ if and only if $(n'-k, m') \in G_m$, and the same is true for $H_m^{(k)}$ and H_m. Now if $m \in \mathbb{N}_{\text{even}}$ is such that $c^{n-k} m \notin \mathbb{N}$ for all $n \in \{0, \dots, k-1\}$ (by assumption this is satisfied in particular for $m = 2$), then

$$\tag{5.5} \begin{aligned} \nu_{\mu^{(k)}}(\{c^{-k} m\}) &= \sum_{n=k}^{\infty} \sum_{m'=1}^{\infty} a_{m'} \delta_{c^{-n} m'}(\{c^{-k} m\}) = \sum_{(n', m') \in G_m^{(k)} \cup H_m^{(k)}} a_{m'} \\ &\le a_m + \sum_{(n', m') \in G_m} a_{m'}. \end{aligned}$$

If G_2 is empty, then (5.5) gives $\nu_{\mu^{(k)}}(\{2c^{-k}\}) \le a_2 < 0$, showing that $\mu^{(k)} \in ID^0$. So suppose that G_2 is non-empty. As shown in [12], this implies $f \ge 1$ and hence β is even, hence α odd, and $\alpha \ne 1$, since $c > 1$. Now choose $m = m_j = 2^{jf}$ for $j \in \mathbb{N}$. Then for each j, as shown in [12], G_{m_j} contains at most one element, and if $G_{m_j} \ne \emptyset$ its unique element (n', m') is given by $n' = j l_0$ and $m' = m_j' = c^{j l_0} m_j$. For $j \in \mathbb{N}$ such that

$$\tag{5.6} c^{n-k} m_j \notin \mathbb{N} \quad \text{for all } n \in \{0, 1, \dots, k-1\},$$

(5.5) gives

$$\nu_{\mu^{(k)}}(\{c^{-k} m_j\}) \le \begin{cases} a_{m_j} < 0, & G_{m_j} = \emptyset, \\ a_{m_j} + a_{m_j'}, & G_{m_j} \ne \emptyset. \end{cases}$$

Since $a_{m_j} + a_{m_j'} < 0$ for $j \in \mathbb{N}$ large enough such that $G_{m_j} \ne \emptyset$ (see [12], p.261), we obtain $\mu^{(k)} \in ID^0$, provided that, for large enough j, condition (5.6) holds. If $c, c^2, \dots, c^k$ are all irrational, then (5.6) is clear. If some of them are rational, then $l_0 \le k$ and $c^{l_0} = \alpha/\beta \ne 2$ and

$$c^{-s l_0} m_j = \beta^s 2^{jf} / \alpha^s \notin \mathbb{N}, \quad \forall\, s \in \mathbb{N}$$

(since $\alpha > 1$ odd), and we have (5.6), recalling that $c^{-n} m_j$ is irrational if n is not an integer multiple of l_0. This finishes the proof of (A).

Let us prove the statement (B). Since $r^2 > p^2 q$ implies $r > pq$, we assume that $q > 0$ and $r > pq$. Then we have (5.2), (5.3), and (5.4). Since we have already proved (A), we consider only the case where $c^l = 2$ for some $l \in \{1, 2, \dots, k\}$. This l is unique. Then it is easy to see that $c^{l'}$ is irrational for $l' = 1, \dots, l-1$. Let s be the largest non-negative integer satisfying $ls \le k-1$. Let $m \in \mathbb{N}_{\mathrm{odd}}$. We have

$$\begin{aligned}\nu_{\mu^{(k)}}(\{c^{-ls}m\}) &= \sum_{n=k}^{\infty}\sum_{m'=1}^{\infty} a_{m'}\delta_{c^{-n}m'}(\{2^{-s}m\}) + \sum_{n=0}^{k-1}\sum_{m'=1}^{\infty}(m')^{-1}q^{m'}\delta_{c^{-n}m'}(\{2^{-s}m\})\\ &= S_1 + S_2.\end{aligned}$$

Recall that m is odd and that $c^{l'}$ is irrational for $l' = 1, \dots, l-1$. Then we see that $S_1 = \sum_{n=1}^{\infty} a_{2^n m}$ and $S_2 = m^{-1}q^m$. Hence we obtain

$$\nu_{\mu^{(k)}}(\{c^{-ls}m\}) < a_{2m} + m^{-1}q^m = (2m)^{-1}(q^{2m} - (r/p)^{2m}) + m^{-1}q^m$$

from (5.4). We conclude that if $\mu^{(k)} \in ID$, then

$$0 < 1 + q^m/2 - (r^2/(p^2q))^m/2, \qquad \forall\, m \in \mathbb{N}_{\mathrm{odd}},$$

which implies $r^2 \le p^2 q$. This finishes the proof of (B). □

The following corollary is now immediate from Theorems 5.2 and 5.3.

Corollary 5.4. *If $k \in \mathbb{N}$, then parameters c, p, q, r exist such that $\mu^{(k)} \in ID$ and $\rho^{(k)} \in ID^0$.*

The following theorem supplements Theorems 4.1 and 4.3.

Theorem 5.5. *Assume $0 < r < p$. Then $\mu^{(k)} \in ID^0$ for all $k \in \mathbb{Z}$ if and only if either* (a) *$r^2 > p^2 q$ or* (b) *$p^2q^2 < r^2 \le p^2 q$ and $c^m \ne 2$ for all $m \in \mathbb{N}$.*

Proof. We have $\mu^{(k)} \in ID \cup ID^0$ by Theorem 4.1. It follows from Theorem 5.3 that $\mu^{(k)} \in ID^0$ for all $k \in \mathbb{N}$ if and only if either (a) or (b) holds. If $\mu^{(k)} \in ID^0$ for all $k \in \mathbb{N}$, then $\mu^{(k)} \in ID^0$ for all $k \in \mathbb{Z}$ by Theorem 4.3. □

The limit distribution of $\mu^{(k)}$ as $k \to \infty$ is as follows.

Theorem 5.6. *Let c, p, q, r be fixed.*

(i) *Assume $q > 0$. Define $(c^\sharp, p^\sharp, q^\sharp, r^\sharp) = (c, 1-q, q, 0)$ and let $\mu^{\sharp(k)}$ be the distribution corresponding to $\mu^{(k)}$ with $(c^\sharp, p^\sharp, q^\sharp, r^\sharp)$ used in place of (c, p, q, r). Then $\mu^{(k)}$ weakly converges to $\mu^{\sharp(0)}$ as $k \to \infty$.*

(ii) *Assume $q = 0$. Then $\mu^{(k)}$ weakly converges to δ_0 as $k \to \infty$.*

We remark that $\mu^{\sharp(k)}$ does not depend on k and is infinitely divisible, so that the limit distribution is infinitely divisible in all cases, although by Theorems 4.1 and 5.5 there are many cases of parameters for which $\mu^{(k)} \notin ID$ for all $k \in \mathbb{Z}$.

Proof. It follows from (2.8) that

$$\widehat{\mu}^{(k)}(z) = \widehat{\mu}^{(0)}(c^{-k}z)\prod_{n=0}^{k-1}\frac{1-q}{1-qe^{ic^{-n}z}}$$

for $k \in \mathbb{N}$. Hence, as $k \to \infty$ we obtain

$$\widehat{\mu}^{(k)}(z) \to \prod_{n=0}^{\infty}\frac{1-q}{1-qe^{ic^{-n}z}} = \begin{cases}\widehat{\mu}^{\sharp(0)}(z), & q > 0,\\ 1 = \widehat{\delta}_0(z), & q = 0.\end{cases}$$

□

Remark. Let the parameters c, p, q, r be fixed and $\{Z_k, k \in \mathbb{Z}\}$ be a sequence of independent identically distributed random variables, geometrically distributed with parameter q if $q > 0$ and distributed as δ_0 if $q = 0$. Let $k_0 \in \mathbb{Z}$ and X_{k_0} be a random variable with distribution $\mu^{(k_0)}$, independent of $\{Z_k, k > k_0\}$. Define $\{X_k, k \geq k_0\}$ inductively by

$$X_{k+1} = c^{-1}X_k + Z_{k+1}, \quad k = k_0, k_0+1, \ldots .$$

Then $\mathcal{L}(X_k) = \mu^{(k)}$ for all $k \geq k_0$ by (2.8), so that the $\mu^{(k)}$ appear naturally as marginal distributions of a certain autoregressive process of order 1. The limit distributions $\mu^{\sharp(0)}$ $(q > 0)$ and δ_0 $(q = 0)$ as $k \to \infty$ described in Theorem 5.6 give the unique stationary distribution of the corresponding AR(1) equation

$$Y_{k+1} = c^{-1}Y_k + Z_{k+1}, \quad k \in \mathbb{Z}.$$

6. Conditions for infinite divisibility for $k < 0$

In this section we obtain necessary and sufficient conditions for $\rho^{(k)} \in ID$ and $\mu^{(k)} \in ID$ when $k < 0$ and then derive some simple consequences of these characterizations. Again, by virtue of Theorem 4.1, we only have to consider the case $0 < r < p$.

Theorem 6.1. *Let k be a negative integer. Assume $0 < r < p$. Then $\rho^{(k)} \in ID$ if and only if $2c^{|k|} \in \mathbb{N}$ and*

$$q^{2c^{|k|}} \geq c^{|k|}(r/p)^2.$$

Proof. The proof of Theorem 5.2 shows that $\mu^{(k)} \in ID$ if and only if $s := 2c^{-k} \in \mathbb{N}$ and (5.1) holds. From $k < 0$ we have $s \geq 3$ and hence

$$q^s \geq (s/2)^{1/m}(r/p)^2, \quad \forall\, m \in \mathbb{N} \quad \Leftrightarrow \quad q^s \geq (s/2)(r/p)^2,$$

which completes the proof. □

The characterization when $\mu^{(k)} \in ID$ for negative k is much more involved and different techniques will be needed according to whether $2c^j \in \mathbb{N}_{\text{even}}$ or $2c^j \in \mathbb{N}_{\text{odd}}$ for some $j \in \mathbb{N}$. In the first case, the characterization will be achieved in terms of the function $h_{\alpha,\gamma}$ defined below. Let $\alpha, \gamma \in \mathbb{N}$ with $\alpha \geq 2$. We use the function

$$x \mapsto F_\alpha(x) = \sum_{n=0}^{\infty} \alpha^{-n} x^{2\alpha^n}, \qquad 0 \leq x \leq 1$$

and the functions $x \mapsto h_{\alpha,\gamma}(x)$ and $x \mapsto f_{\alpha,\gamma}(x)$ for $0 < x \leq 1$ defined by the relations

$$\begin{aligned} \alpha^{-\gamma} F_\alpha(x) &= F_\alpha(h_{\alpha,\gamma}(x)), \\ f_{\alpha,\gamma}(x) &= x^{-1} h_{\alpha,\gamma}(x). \end{aligned} \tag{6.1}$$

Observe that F_α is strictly increasing and continuous on $[0,1]$ with $F_\alpha(0) = 0$ and hence $h_{\alpha,\gamma}(x)$ is uniquely definable for $x \in (0,1]$ and it holds $0 < h_{\alpha,\gamma}(x) < x$. The next proposition describes some properties of $h_{\alpha,\gamma}$ which will be used in the sequel.

Proposition 6.2. *The functions $h_{\alpha,\gamma}$ and $f_{\alpha,\gamma}$ are continuous and strictly increasing on $(0,1]$ and satisfy*

$$\lim_{x \downarrow 0} f_{\alpha,\gamma}(x) = \alpha^{-\gamma/2}, \tag{6.2}$$

$$f_{\alpha,\gamma}(1) = h_{\alpha,\gamma}(1) < \alpha^{-\gamma/4}, \tag{6.3}$$

$$h_{\alpha,\gamma}(1) < \alpha^{-\gamma/2}(1 + \alpha^{-1}) \quad \textit{for all } \gamma \textit{ if } \alpha \textit{ is large enough}, \tag{6.4}$$

$$h_{\alpha,\gamma}(x) > h_{\alpha,\gamma+1}(x), \qquad \forall\, x \in (0,1], \tag{6.5}$$

$$h_{\alpha,\gamma}(x^n) \geq (h_{\alpha,\gamma}(x))^n, \qquad \forall\, x \in (0,1] \;\; \forall\, n \in \mathbb{N}. \tag{6.6}$$

Proof. Since F_α is a continuous strictly increasing function defined on $[0,1]$, it follows that $h_{\alpha,\gamma}$ is continuous and strictly increasing on $(0,1]$, and hence that $f_{\alpha,\gamma}$ is continuous. Also observe that $h_{\alpha,\gamma}(x) \to 0$ as $x \downarrow 0$ since $F_\alpha(0) = 0$. From (6.1) and the Taylor expansion of F_α we obtain as $x \downarrow 0$,

$$\alpha^{-\gamma} x^2 \bigl(1 + O(x^{2(\alpha-1)})\bigr) = (h_{\alpha,\gamma}(x))^2 \bigl(1 + O((h_{\alpha,\gamma}(x))^{2(\alpha-1)})\bigr)$$

from which (6.2) follows. In order to show (6.3), first let us check that

$$F_\alpha(\alpha^{-\gamma/4}) > \alpha^{1-\gamma}/(\alpha - 1). \tag{6.7}$$

Indeed, if $\alpha \geq 3$ or $\gamma \geq 2$, then use $F_\alpha(x) > x^2$ and obtain

$$F_\alpha(\alpha^{-\gamma/4}) > \alpha^{-\gamma/2} = \alpha^{1-\gamma} \alpha^{\gamma/2-1} > \alpha^{1-\gamma}/(\alpha - 1),$$

since $\alpha^{\gamma/2-1} \geq 1/(\alpha-1)$ if $\alpha \geq 3$ or $\gamma \geq 2$. If $\alpha = 2$ and $\gamma = 1$, then use $F_2(x) > x^2 + x^4/2 + x^8/4$ to obtain

$$F_2(2^{-1/4}) > 2^{-1/2} + 2^{-2} + 2^{-4} = 0.7071\cdots + 0.25 + 0.0625 = 1.0196\cdots > 1,$$

which proves (6.7). Since $F_\alpha(1) = \alpha/(\alpha-1)$, we have $F_\alpha(h_{\alpha,\gamma}(1)) = \alpha^{1-\gamma}/(\alpha-1)$. Hence (6.3) follows from (6.7).

To see (6.4) it is enough to show that

$$F_\alpha(\alpha^{-\gamma/2}(1+\alpha^{-1})) > \alpha^{1-\gamma}/(\alpha-1) \quad \text{for all } \gamma \text{ if } \alpha \text{ is large enough.} \tag{6.8}$$

Since $F_\alpha(x) > x^2$, we have

$$F_\alpha(\alpha^{-\gamma/2}(1+\alpha^{-1})) > \alpha^{-\gamma}(1+\alpha^{-1})^2 > \alpha^{-\gamma}(1+2\alpha^{-1}).$$

On the other hand,

$$\alpha^{1-\gamma}/(\alpha-1) = \alpha^{-\gamma}(1+\alpha^{-1}+O(\alpha^{-2})), \quad \alpha \to \infty.$$

Hence (6.8) holds.

To see (6.5), observe that

$$F_\alpha(h_{\alpha,\gamma}(x)) = \alpha^{-\gamma}F_\alpha(x) = \alpha\, F_\alpha(h_{\alpha,\gamma+1}(x))$$

by (6.1), which together with the strict increase of F_α implies (6.5).

Before we can prove (6.6), we need to show that $f_{\alpha,\gamma}$ is strictly increasing. For that, let us first show that

$$f_{\alpha,\gamma}(x) < f_{\alpha,\gamma}(1), \quad \forall\, x \in (0,1). \tag{6.9}$$

Suppose, on the contrary, that $f_{\alpha,\gamma}(x_0) \geq f_{\alpha,\gamma}(1)$ for some $x_0 \in (0,1)$. Then $\alpha^{-\gamma}F_\alpha(x_0) = F_\alpha(f_{\alpha,\gamma}(x_0)x_0) \geq F_\alpha(f_{\alpha,\gamma}(1)x_0)$, that is,

$$\sum_{n=0}^{\infty} \alpha^{-n}(\alpha^{-\gamma} - f_{\alpha,\gamma}(1)^{2\alpha^n})x_0^{2\alpha^n} \geq 0.$$

Let

$$G_{\alpha,\gamma}(\xi) = \sum_{n=0}^{\infty} \alpha^{-n}(\alpha^{-\gamma} - f_{\alpha,\gamma}(1)^{2\alpha^n})\xi^{2(\alpha^n-1)}, \quad \xi \in [0,1].$$

Then $G_{\alpha,\gamma}(x_0) \geq 0$ and $G_{\alpha,\gamma}(1) = 0$, which follows from $\alpha^{-\gamma}F_\alpha(1) = F_\alpha(f_{\alpha,\gamma}(1))$. But we have

$$G'_{\alpha,\gamma}(\xi) = \sum_{n=1}^{\infty} 2(\alpha^n-1)\alpha^{-n}(\alpha^{-\gamma} - f_{\alpha,\gamma}(1)^{2\alpha^n})\xi^{2(\alpha^n-1)-1} > 0, \quad \xi \in (0,1), \tag{6.10}$$

since

$$\alpha^{-\gamma} - f_{\alpha,\gamma}(1)^{2\alpha^n} \geq \alpha^{-\gamma} - f_{\alpha,\gamma}(1)^{2\alpha} > \alpha^{-\gamma} - \alpha^{-\alpha\gamma/2} \geq 0$$

for $n \geq 1$ by (6.3). This is absurd. Hence (6.9) is true.

Now we show that $f_{\alpha,\gamma}$ is strictly increasing on $(0,1]$. Suppose that there exist x_1 and x_2 in $(0,1]$ such that $x_1 < x_2$ and $f_{\alpha,\gamma}(x_1) \geq f_{\alpha,\gamma}(x_2)$. Then

$$\alpha^{-\gamma} F_\alpha(x_1) = F_\alpha(f_{\alpha,\gamma}(x_1)x_1) \geq F_\alpha(f_{\alpha,\gamma}(x_2)x_1),$$
$$\alpha^{-\gamma} F_\alpha(x_2) = F_\alpha(f_{\alpha,\gamma}(x_2)x_2),$$

that is,

$$\alpha^{-\gamma}\sum_{n=0}^{\infty}\alpha^{-n}x_1^{2\alpha^n} \geq \sum_{n=0}^{\infty}\alpha^{-n}f_{\alpha,\gamma}(x_2)^{2\alpha^n}x_1^{2\alpha^n},$$
$$\alpha^{-\gamma}\sum_{n=0}^{\infty}\alpha^{-n}x_2^{2\alpha^n} = \sum_{n=0}^{\infty}\alpha^{-n}f_{\alpha,\gamma}(x_2)^{2\alpha^n}x_2^{2\alpha^n}.$$

Define

$$H_{\alpha,\gamma}(\xi) = \sum_{n=0}^{\infty}\alpha^{-n}(\alpha^{-\gamma} - f_{\alpha,\gamma}(x_2)^{2\alpha^n})\xi^{2(\alpha^n-1)}, \quad \xi \in [0,1].$$

Then we have $H_{\alpha,\gamma}(x_1) \geq 0$ and $H_{\alpha,\gamma}(x_2) = 0$. On the other hand, noting that $f_{\alpha,\gamma}(x_2) \leq f_{\alpha,\gamma}(1)$ by (6.9), we can prove $H'_{\alpha,\gamma}(\xi) > 0$ in the same way as the proof of (6.10). This is a contradiction. Hence $f_{\alpha,\gamma}$ is strictly increasing.

Finally, (6.6) is proved. Indeed, this is trivial for $n = 1$, and for $n \geq 2$ we have

$$(h_{\alpha,\gamma}(x))^n = (f_{\alpha,\gamma}(x))^n x^n < \alpha^{-n\gamma/4}x^n \leq \alpha^{-\gamma/2}x^n < f_\alpha(x^n)x^n = h_{\alpha,\gamma}(x^n),$$

noting that $f_{\alpha,\gamma}(x)$ is strictly increasing and using (6.2) and (6.3). $\square$

Now we give the classification when $\mu^{(k)} \in ID$ for $k < 0$ and $0 < r < p$. As usual, for $x \in \mathbb{R}$ we shall denote by $\lfloor x \rfloor$ the largest integer being smaller than or equal to x, and by $\lceil x \rceil$ the smallest integer being greater than or equal to x.

Theorem 6.3. *Let k be a negative integer. Assume $0 < r < p$.*

(i) *If $2c^j \notin \mathbb{N}$ for all integers j satisfying $j \geq |k|$, then $\mu^{(k)} \in ID^0$.*

(ii) *Suppose that $c^j \in \mathbb{N}$ for some $j \in \mathbb{N}$. Let l be the smallest of such j and let $\alpha = c^l$, $\beta := \lceil |k|/l \rceil$ and $h_{\alpha,\beta}$ be defined by (6.1). Then $\mu^{(k)} \in ID$ if and only if $q > 0$ and*

$$h_{\alpha,\beta}(q^{\alpha^\beta}) \geq r/p. \tag{6.11}$$

(iii) *Suppose that $2c^j \in \mathbb{N}_{\text{odd}}$ for some $j \in \mathbb{N}$ with $j \geq |k|$. Then $c^{j'} \notin \mathbb{N}$ for all $j' \in \mathbb{N}$, and $j \in \mathbb{N}$ satisfying $2c^j \in \mathbb{N}_{\text{odd}}$ is unique. Let $\alpha = c^j$. Then $2\alpha \in \mathbb{N}_{\text{odd}}$ and $2\alpha \geq 3$.*

(iii)$_1$ *Suppose that* $2\alpha \geq 5$. *Then* $\mu^{(k)} \in ID$ *if and only if*

$$q^{2\alpha} + (r/p)^{2\alpha} \geq \alpha\,(r/p)^2. \tag{6.12}$$

(iii)$_2$ *Suppose that* $2\alpha = 3$. *For* $m \in \mathbb{N}$, *denote by* $t(m)$ *the largest integer* t' *such that* m *is an integer multiple of* $2^{t'}$, *and write* $a_m := m^{-1}(q^m - (-r/p)^m)$ *Then* $\mu^{(k)} \in ID$ *if and only if*

$$\sum_{s=0}^{t(m)} a_{3^{s+1}2^{-s}m} \geq (2m)^{-1}(r/p)^{2m}, \quad \forall\, m \in \{1, \ldots, 149\}. \tag{6.13}$$

Proof. For all cases (i) – (iii) observe that we have (4.3) and (4.4) since $0 < r < p$. Therefore

$$\nu_{\mu^{(k)}} = \sum_{n=0}^{\infty}\sum_{m'=1}^{\infty} a_{m'}\delta_{c^{-n}m'} + \sum_{s'=1}^{|k|}\sum_{m'=1}^{\infty}(m')^{-1}(-1)^{m'+1}(r/p)^{m'}\delta_{c^{s'}m'} \tag{6.14}$$

with

$$a_{m'} = (m')^{-1}(q^{m'} - (-r/p)^{m'}). \tag{6.15}$$

We have $\mu^{(k)} \in ID$ if and only if $\nu_{\mu^{(k)}} \geq 0$.

To prove (i), assume that $2c^j \notin \mathbb{N}$ for $j \geq |k|$. Consider $\nu_{\mu^{(k)}}(\{2c^{|k|}\})$. Let $E = \{s \in \{1, 2, \cdots, |k|-1\} \colon 2c^s \in \mathbb{N}_{\mathrm{odd}}\}$. Since $2c^j \notin \mathbb{N}$ for $j \geq |k|$, (6.14) gives

$$\nu_{\mu^{(k)}}(\{2c^{|k|}\}) \leq -(r/p)^2/2 + \sum_{s\in E}(2c^s)^{-1}(r/p)^{2c^s}.$$

But since E contains at most one element, we have

$$\sum_{s\in E}(2c^s)^{-1}(r/p)^{2c^s} < (r/p)^2/2,$$

so that $\nu_{\mu^{(k)}}(\{2c^{|k|}\}) < 0$. Hence $\mu^{(k)} \in ID^0$.

Let us prove (ii). Assume that $c^j \in \mathbb{N}$ for some $j \in \mathbb{N}$ and let l, α, β be as in the statement of the theorem. If $q = 0$ or if $q > 0$ and $r > pq$, then $\mu^{(0)} \notin ID$ and hence $\mu^{(k)} \notin ID$ by Proposition 5.1 and Theorem 4.3. Since $h_{\alpha,\beta}(q^{\alpha^\beta}) < q^{\alpha^\beta} < q$ for $q > 0$, condition (6.11) implies $r \leq pq$. Hence we may assume $q > 0$ and $r \leq pq$ from now on, which in particular implies $a_{m'} \geq 0$. Hence

$$\mu^{(k)} \in ID \quad \Leftrightarrow \quad \begin{array}{l}\nu_{\mu(k)}(\{z\}) \geq 0 \text{ for all } z \text{ of the form } z = 2mc^s \\ \text{with } s \in \{1, \ldots, |k|\},\ m \in \mathbb{N}.\end{array} \tag{6.16}$$

For $s \in \{1, \ldots, |k|\}$ and $m \in \mathbb{N}$ denote

$$g(s,m) := \sum_{n\in\mathbb{N}_0 : 2mc^{s+n}\in\mathbb{N}} a_{2mc^{s+n}} - \sum_{n\in\{0,\ldots,s-1\} : 2mc^n\in\mathbb{N}} (2m)^{-1}c^{-n}(r/p)^{2mc^n}.$$

If $z = 2mc^s$ with $s \in \{1, \dots, |k| - 1\}$ and $m \in \mathbb{N}$, but cannot be written in the form $z = 2m'c^{s'}$ with $m' \in \mathbb{N}$ and $s' \in \{s + 1, \dots, |k|\}$, then $2mc^{s-s'} \notin \mathbb{N}_{\text{even}}$ for $s' \in \{s + 1, \dots, |k|\}$, and hence by (6.14)

$$\begin{aligned} &\nu_{\mu^{(k)}}(\{z\}) \\ (6.17)\qquad &\geq \sum_{n\in\mathbb{N}_0: 2mc^{s+n}\in\mathbb{N}} a_{2mc^{s+n}} - \sum_{s'\in\{1,\dots,s\}: 2mc^{s-s'}\in\mathbb{N}} \frac{1}{2mc^{s-s'}} (r/p)^{2mc^{s-s'}} = g(s, m). \end{aligned}$$

If $z = 2mc^{|k|}$ with $m \in \mathbb{N}$, then

$$\begin{aligned} &\nu_{\mu^{(k)}}(\{z\}) \\ (6.18)\qquad &= \sum_{n\in\mathbb{N}_0: 2mc^{|k|+n}\in\mathbb{N}} a_{2mc^{|k|+n}} - \sum_{s'\in\{1,\dots,|k|\}: 2mc^{|k|-s'}\in\mathbb{N}} \frac{1}{2mc^{|k|-s'}} (r/p)^{2mc^{|k|-s'}} \\ &= g(|k|, m). \end{aligned}$$

We claim that for $s \in \{1, \dots, |k|\}$ we have

$$(6.19)\qquad g(s, m) \geq 0, \quad \forall\, m \in \mathbb{N} \quad \Leftrightarrow \quad h_{\alpha,\lceil s/l\rceil}(q^{\alpha^{\lceil s/l\rceil}}) \geq r/p.$$

Once we have established (6.19), then (6.16) and (6.18) show that (6.11) is necessary for $\mu^{(k)} \in ID$, while (6.16) – (6.18) show that it is also sufficient, since monotonicity of $h_{\alpha,\lceil s/l\rceil}$ and (6.5) imply

$$h_{\alpha,\lceil s/l\rceil}(q^{\alpha^{\lceil s/l\rceil}}) \geq h_{\alpha,\lceil s/l\rceil}(q^{\alpha^\beta}) \geq h_{\alpha,\beta}(q^{\alpha^\beta}), \qquad \forall\, s \in \{1, \dots, |k|\}.$$

To show (6.19), observe that for $j \in \mathbb{N}_0$ we have $c^j = \alpha^{j/l} \in \mathbb{N}$ if and only if j is an integer multiple of l, and that c^j is irrational otherwise. From this property, we see that

$$g(s, m) = \sum_{n'=0}^{\infty} a_{2m\alpha^{\lceil s/l\rceil}\alpha^{n'}} - \sum_{n'=0}^{\lfloor (s-1)/l\rfloor} (2m)^{-1}\alpha^{-n'}(r/p)^{2m\alpha^{n'}}.$$

Observing that $\lfloor (s - 1)/l\rfloor = \lceil s/l\rceil - 1$, we have for $s \in \{1, \dots, |k|\}$ and $m \in \mathbb{N}$,

$$\begin{aligned} &g(s, m) \geq 0 \\ \Leftrightarrow\quad &\sum_{n=0}^{\infty} \frac{1}{2m\alpha^{\lceil s/l\rceil+n}} \left[q^{2m\alpha^{\lceil s/l\rceil+n}} - (r/p)^{2m\alpha^{\lceil s/l\rceil+n}}\right] \geq \sum_{n=0}^{\lceil s/l\rceil-1} \frac{1}{2m\alpha^n}(r/p)^{2m\alpha^n} \\ \Leftrightarrow\quad &\alpha^{-\lceil s/l\rceil} \sum_{n=0}^{\infty} \alpha^{-n}(q^{m\alpha^{\lceil s/l\rceil}})^{2\alpha^n} \geq \sum_{n=0}^{\infty} \alpha^{-n}\left[(r/p)^m\right]^{2\alpha^n} \\ \Leftrightarrow\quad &\alpha^{-\lceil s/l\rceil} F_\alpha(q^{m\alpha^{\lceil s/l\rceil}}) \geq F_\alpha((r/p)^m) \\ \Leftrightarrow\quad &h_{\alpha,\lceil s/l\rceil}(q^{m\alpha^{\lceil s/l\rceil}}) \geq (r/p)^m. \end{aligned}$$

Now (6.19) follows from property (6.6) of $h_{\alpha,\lceil s/l\rceil}$, completing the proof of (ii).

Let us prove (iii). Assume that $2c^{j''} \in \mathbb{N}_{\text{odd}}$ for some $j'' \in \mathbb{N}$ with $j'' \geq |k|$. Let j be the smallest positive integer such that $c^j \in \mathbb{Q}$. Then $c^{j'} \in \mathbb{Q}$ with $j' \in \mathbb{N}$ if and only if j' is an integer multiple of j. We have $c^j \notin \mathbb{N}$, since $2c^{j''} \in \mathbb{N}_{\text{odd}}$ for some $j'' \in \mathbb{N}$. Denote $c^j = a'/b'$ with $a', b' \in \mathbb{N}$ having no common divisor. Then $2c^{nj} = 2(a'/b')^n \notin \mathbb{N}$ for all $n \in \mathbb{N}$ with $n \geq 2$. Hence $c^{j'} \notin \mathbb{N}$ for all $j' \in \mathbb{N}$ and $2c^{j'} \in \mathbb{N}_{\text{odd}}$ if and only if $j' = j$, so that $j = j''$. As in the proof of (ii), if $q = 0$, or if $q > 0$ and $r > pq$, then $\mu^{(k)} \notin ID$. On the other hand, (6.12) for $2\alpha \geq 5$ clearly implies $r \leq pq$, and as will be shown later in Equation (6.27) in the proof of $(\text{iii})_2$, (6.13) implies $q^3 \geq (r/p)^2(5/4)^{1/75}$. Hence we may and do assume $q > 0$ and $r \leq pq$ from now on (this assumption will not be needed when (6.27) is derived from (6.13) in $(\text{iii})_2$). In particular, $a_{m'} \geq 0$ for $m' \in \mathbb{N}$, and $\mu^{(k)} \in ID$ is characterized by the right-hand side of (6.16). But as follows from the discussion above and the fact that $j \geq |k|$, if $z = 2mc^s$ with $s \in \{1, \ldots, |k|\}$ and $m \in \mathbb{N}$, then $z = c^{s'}m'$ for $s' \in \{1, \ldots, |k|\}$ and $m' \in \mathbb{N}$ if and only if $s' = s$ and $m' = 2m$. Further, since $c^{s+n} \in \mathbb{Q}$ for $s \in \{1, \ldots, |k|\}$ and $n \in \mathbb{N}_0$ if and only if $s + n = (s' + 1)j$ for some $s' \in \mathbb{N}_0$, in which case $c^{s+n} = \alpha^{s'+1}$, (6.14) gives

$$\nu_{\mu^{(k)}}(\{2mc^s\}) = \sum_{s' \in \mathbb{N}_0 : 2m\alpha^{s'+1} \in \mathbb{N}} a_{2m\alpha^{s'+1}} - (2m)^{-1}(r/p)^{2m} \tag{6.20}$$

for $s \in \{1, \ldots, |k|\}$ and $m \in \mathbb{N}$. Observe that this quantity does not depend on s. Denote by $t(m)$ the largest integer t' such that m is an integer multiple of $2^{t'}$, and observe that $2m\alpha^{s'+1} = (2\alpha)^{s'+1}2^{-s'}m \in \mathbb{N}$ with $s' \in \mathbb{N}_0$ if and only if $s' \leq t(m)$ due to the assumption $2\alpha \in \mathbb{N}_{\text{odd}}$. From (6.16) and (6.20) we hence conclude that

$$\mu^{(k)} \in ID \quad \Leftrightarrow \quad \sum_{s=0}^{t(m)} a_{2m\alpha^{s+1}} \geq (2m)^{-1}(r/p)^{2m} \text{ for all } m \in \mathbb{N}. \tag{6.21}$$

$(\text{iii})_1$ Now assume that $b := 2\alpha \geq 5$. If $\mu^{(k)} \in ID$, then (6.21) with $m = 1$ gives (6.12), so that (6.12) is necessary for $\mu^{(k)} \in ID$. To show that it is also sufficient, assume that (6.12) holds for the rest of the proof of $(\text{iii})_1$. We first claim that we have

$$q^b \geq (r/p)^b + (r/p)^2. \tag{6.22}$$

Indeed, if $b \geq 7$, then (6.12) gives

$$q^b > (\frac{b}{2} - 1)(r/p)^2 \geq 2(r/p)^2 \geq (r/p)^b + (r/p)^2,$$

which is (6.22). If $b = 5$, consider the function $\varphi : \mathbb{R} \to \mathbb{R}$, $x \mapsto (5/2)x^2 - x^5$. Then $\varphi'(x) = 5x(1 - x^3)$, so that φ is increasing on $[0, 1]$. But $\varphi\left((3/4)^{1/3}\right) \approx 1.4 > 1$, so that $(5/2)(r/p)^2 - (r/p)^5 > 1$ whenever $r/p \geq (3/4)^{1/3}$. But since $q < 1$, it follows that (6.12) for $b = 5$ implies $r/p \leq (3/4)^{1/3}$, and hence (6.12) gives

$$q^5 - \left[(r/p)^5 + (r/p)^2\right] \geq (3/2)(r/p)^2 - 2(r/p)^5 = (3/2)(r/p)^2\left[1 - (4/3)(r/p)^3\right] \geq 0,$$

which is (6.22) also in the case $b = 5$.

Denote $d := b\log(qp/r)$, $\delta := (b-2)\log(p/r)$, and define the functions $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ by $\varphi_1(x) = e^{dx} + 1$ and $\varphi_2(x) = (b/2)e^{\delta x}$, respectively. Observe that $\varphi_1(0) = 2 < (b/2) = \varphi_2(0)$. Further, (6.12) gives $\varphi_1(1) \geq \varphi_2(1)$, and (6.22) shows that $d > \delta$. Let $x_0 \in (0, 1]$ be the point satisfying $\varphi_1(x_0) = \varphi_2(x_0)$ and $\varphi_1(x) < \varphi_2(x)$ for $x \in [0, x_0)$. Then $\varphi_1'(x_0) \geq \varphi_2'(x_0)$, that is, $de^{(d-\delta)x_0} \geq \delta b/2$, which implies $de^{(d-\delta)x} > \delta b/2$ for $x > x_0$. Hence, for all $x > x_0$, $\varphi_1'(x) > \varphi_2'(x)$ and $\varphi_1(x) > \varphi_2(x)$. This gives

$$\left(\frac{qp}{r}\right)^{bm} + 1 \geq \frac{b}{2}\,(p/r)^{(b-2)m}, \quad \forall\, m \in \mathbb{N}_{\text{odd}},$$

so that $a_{bm} \geq (2m)^{-1}(r/p)^{2m}$ for $m \in \mathbb{N}_{\text{odd}}$ which is the right-hand side of (6.21) for $m \in \mathbb{N}_{\text{odd}}$. Hence it only remains to show that the right-hand side of (6.21) holds for all $m \in \mathbb{N}_{\text{even}}$, too. Since $\sum_{s=0}^{t(m)} a_{2m\alpha^{s+1}} \geq a_{2m\alpha} = a_{bm}$, by induction it is enough to prove the following for $m \in \mathbb{N}$:

(6.23) If $a_{bm} \geq (2m)^{-1}(r/p)^{2m}$, then $a_{2bm} \geq (4m)^{-1}(r/p)^{4m}$.

So assume that $a_{bm} \geq (2m)^{-1}(r/p)^{2m}$. If $m \in \mathbb{N}_{\text{even}}$, this means that $q^{bm} - (r/p)^{bm} \geq (b/2)(r/p)^{2m}$, and it follows that

$$\begin{aligned} a_{2bm} &= \frac{1}{2bm}\left[q^{bm} - (r/p)^{bm}\right]\left[q^{bm} + (r/p)^{bm}\right] \\ &\geq \frac{1}{4m}(r/p)^{2m}\left[q^{bm} + (r/p)^{bm}\right] \\ &\geq \frac{1}{4m}(r/p)^{2m}(r/p)^{2m} = \frac{1}{4m}(r/p)^{4m}, \end{aligned}$$

where the last inequality follows from (6.22). If $m \in \mathbb{N}_{\text{odd}}$, then $a_{bm} \geq (2m)^{-1}(r/p)^{2m}$ means that $q^{bm} + (r/p)^{bm} \geq (b/2)(r/p)^{2m}$, so that

$$\begin{aligned} a_{2bm} &= \frac{1}{2bm}\left[q^{bm} + (r/p)^{bm}\right]\left[q^{bm} - (r/p)^{bm}\right] \\ &\geq \frac{1}{4m}(r/p)^{2m}\left[q^{bm} - (r/p)^{bm}\right] \\ &\geq \frac{1}{4m}(r/p)^{4m}, \end{aligned}$$

where the last inequality follows from the fact that (6.22) implies

$$q^b \geq \left[(r/p)^{bm} + (r/p)^{2m}\right]^{1/m}.$$

This completes the proof of (6.23), and it follows that (6.12) implies the right-hand side of (6.21) for all $m \in \mathbb{N}$, so that (6.12) is also sufficient for $\mu^{(k)} \in ID$.

$(\text{iii})_2$ Now assume that $b := 2\alpha = 3$. Clearly, (6.13) is nothing else than the right-hand side of (6.21) for $m \leq 149$, showing that (6.13) is necessary for $\mu^{(k)} \in ID$. For the converse, assume that (6.13) holds (without assuming a priori that $r \leq pq$). Then (6.13) applied with $m = 2$ gives $a_6 + a_9 \geq 4^{-1}(r/p)^4$, which is equivalent to

$$q^6 + (2/3)q^9 \geq (3/2)(r/p)^4 + (r/p)^6 - (2/3)(r/p)^9.$$

But since $q < 1$ and $r/p < 1$, this implies

$$(6.24) \quad 5/3 \geq (r/p)^4 \left[3/2 + (r/p)^2 - (2/3)(r/p)^5\right] \geq (r/p)^4 \left[3/2 + (r/p)^2 - 2/3\right].$$

Using $x^4\,[5/6 + x^2] \geq 1.6686... > 5/3$ for $x \geq (13/14)^{1/4}$, (6.24) gives

$$(6.25) \qquad r/p < (13/14)^{1/4}.$$

Applying (6.13) with $m = 75$, i.e. using $a_{225} \geq 150^{-1}(r/p)^{150}$, gives

$$(6.26) \qquad q^3 \geq (r/p)^2 \left[3/2 - (r/p)^{75}\right]^{1/75}.$$

An application of (6.25) shows that

$$\left[3/2 - (r/p)^{75}\right]^{1/75} \geq \left[3/2 - (13/14)^{75/4}\right]^{1/75} = [3/2 - 0.2491...]^{1/75} \geq (5/4)^{1/75},$$

which together with (6.26) results in

$$(6.27) \qquad q^3 \geq (r/p)^2 (5/4)^{1/75}.$$

Now if $m \geq 150$, it follows from (6.25) that

$$\begin{aligned} [3/2 + (r/p)^m]^{1/m} &\leq \left[3/2 + (13/14)^{m/4}\right]^{1/m} \\ &\leq \left[3/2 + (13/14)^{150/4}\right]^{1/150} = (1.2498...)^{1/75} < (5/4)^{1/75}. \end{aligned}$$

Together with (6.27) this shows that

$$q^3 \geq (r/p)^2\,[3/2 + (r/p)^m]^{1/m}, \quad m \geq 150.$$

But for m even, $m \geq 150$, the last equation is equivalent to $a_{3m} \geq (2m)^{-1}(r/p)^{2m}$. On the other hand, if m is odd and $m \geq 150$, then (6.27) gives

$$a_{3m} = \frac{1}{3m}\left[q^{3m} + (r/p)^{3m}\right] \geq \frac{1}{3m} q^{3m} \geq \frac{1}{3m}(r/p)^{2m}(5/4)^{m/75} \geq \frac{1}{2m}(r/p)^{2m},$$

where we used $(5/4)^2 \geq 3/2$ in the last inequality. Hence we obtain for $m \in \mathbb{N}$, $m \geq 150$, that

$$\sum_{s=0}^{t(m)} a_{3^{s+1}2^{-s}m} \geq a_{3m} \geq (2m)^{-1}(r/p)^{2m},$$

so that (6.13) implies the right-hand side of (6.21). Hence (6.13) is sufficient for $\mu^{(k)} \in ID$, completing the proof. $\square$

Remark. The condition (6.12) in Theorem 6.3 means $a_{2\alpha} \geq 2^{-1}(r/p)^2$ for a_m of (6.15), which together with $j \geq |k|$ completely characterizes when $\mu^{(k)} \in ID$ in the case of $(\text{iii})_1$ when $2\alpha \geq 5$. This is different in the case $2\alpha = 3$ of $(\text{iii})_2$. Here, the condition $a_{2\alpha} \geq 2^{-1}(r/p)^2$ is not enough to ensure that $\mu^{(k)} \in ID$. For example, if $q^3 > 1/2$ and $r/p = (13/14)^{1/4}$, then $a_{2\alpha} \geq 2^{-1}(r/p)^2$, which is (6.13) for $m = 1$, but $\mu^{(k)} \notin ID$, since (6.13) for $m = 2$ implies (6.25) as shown in the proof of $(\text{iii})_2$. Nevertheless, there seems room to reduce the 149 conditions of (6.13) to a smaller number, but we shall not investigate this subject further.

The following corollary gives handy sufficient and handy necessary conditions for $\mu^{(k)} \in ID$.

Corollary 6.4. *Let k be a negative integer and assume that $0 < r < p$.*

(i) Suppose that $c^j \in \mathbb{N}$ for some $j \in \mathbb{N}$. Let l be the smallest of such j and let $\alpha = c^l$ and $\beta := \lceil |k|/l \rceil$. Then $q^{\alpha^\beta} > \alpha^{\beta/4}(r/p)$ is a necessary condition for $\mu^{(k)} \in ID$, while $q^{\alpha^\beta} \geq \alpha^{\beta/2}(r/p)$ is a sufficient condition for $\mu^{(k)} \in ID$.

(ii) Suppose that $2c^j \in \mathbb{N}_{\text{odd}}$ for some $j \in \mathbb{N}$ with $j \geq |k|$, and let $\alpha = c^j$. Then $q^\alpha > r/p$ is a necessary condition for $\mu^{(k)} \in ID$, and $q^\alpha \geq \alpha^{1/2}(r/p)$ is a sufficient condition for $\mu^{(k)} \in ID$. If $2\alpha \geq 5$, then $q^\alpha > (\alpha - 1)^{1/2}(r/p)$ is another necessary condition for $\mu^{(k)} \in ID$.

Proof. To prove (i), observe that $f_{\alpha,\beta}$ is strictly increasing by Proposition 6.2, so that

$$\alpha^{-\beta/2} < q^{-\alpha^\beta} h_{\alpha,\beta}(q^{\alpha^\beta}) < \alpha^{-\beta/4}$$

for $q \in (0, 1)$ by (6.2) and (6.3). The assertion now follows from (6.11).

To prove (ii), observe that by (6.21) a necessary condition for $\mu^{(k)} \in ID$ is that $a_{2\alpha m} \geq (2m)^{-1}(r/p)^{2m}$ for all $m \in \mathbb{N}_{\text{odd}}$. The latter condition is equivalent to

$$\left[q^{2\alpha}(p/r)^2\right]^m + (r/p)^{(2\alpha-2)m} \geq \alpha, \qquad \forall\, m \in \mathbb{N}_{\text{odd}},$$

which shows that $q^{2\alpha} > (r/p)^2$ is a necessary condition for $\mu^{(k)} \in ID$ by letting m tend to infinity. It is immediate from (6.12) that $q^\alpha > (\alpha - 1)^{1/2}(r/p)$ is necessary for $\mu^{(k)} \in ID$ if $2\alpha \geq 5$. If $2\alpha \geq 3$, then $j \geq |k|$ and $q^\alpha \geq \alpha^{1/2}(r/p)$ imply

$$q^{2c^{|k|}} \geq q^{2c^j} = q^{2\alpha} \geq \alpha(r/p)^2 \geq c^{|k|}(r/p)^2,$$

so that $q^\alpha \geq \alpha^{1/2}(r/p)$ is a sufficient condition for $\rho^{(k)} \in ID$ by Theorem 6.1 and hence for $\mu^{(k)} \in ID$. $\square$

Remark. In the case of Corollary 6.4 (i), another necessary condition for $\mu^{(k)} \in ID$ is that $q^{\alpha^\beta} > \alpha^{\beta/2}(1+\alpha^{-1})^{-1}(r/p)$, provided α is large enough. The proof is the same but using (6.4) instead of (6.3). Compare with the sufficient condition $q^{\alpha^\beta} \geq \alpha^{\beta/2}(r/p)$.

The following corollary is immediate from Theorems 4.1, 6.1 and 6.3.

Corollary 6.5. *If k is a negative integer, then parameters c, p, q, r exist such that $\mu^{(k)} \in ID$ and $\rho^{(k)} \in ID^0$.*

The following Theorem complements Theorems 4.3 and 5.5.

Theorem 6.6. *Let $c > 1$ and p, q, r be fixed such that $p, r > 0$ and $p \neq r$. Then there is $k_0 \in \mathbb{Z}$ such that $\mu^{(k)} \in ID^0$ for all $k \in \mathbb{Z}$ with $k < k_0$.*

Proof. By Theorem 4.1 it only remains to consider the case $0 < r < p$. Since a sequence $\{j_k, k \in \mathbb{N}\}$ of integers tending to ∞ such that $2c^{j_k} \in \mathbb{N}$ for all k can only exist if $c^j \in \mathbb{N}$ for some $j \in \mathbb{N}$, Theorem 6.3 (i) gives the assertion unless $c^j \in \mathbb{N}$ for some $j \in \mathbb{N}$. In the latter case, let α and l be defined defined as in Theorem 6.3 (ii) and $\beta_k := \lceil |k|/l \rceil$. Then $\beta_k \to \infty$ as $k \to -\infty$ and hence $h_{\alpha,\beta_k}(q^{\alpha^{\beta_k}}) \to 0$ as $k \to -\infty$ by (6.1). In particular, (6.11) is violated for large enough $|k|$. ☐

7. Symmetrizations

In general, the symmetrization σ^{sym} of a distribution σ is defined to be the distribution with characteristic function $|\widehat{\sigma}(z)|^2$. It is clear that

$$\text{if } \sigma \in ID, \text{ then } \sigma^{\mathrm{sym}} \in ID. \tag{7.1}$$

It follows from (1.7) that

$$\widehat{\mu}^{(k)\,\mathrm{sym}}(z) = \widehat{\rho}^{(k)\,\mathrm{sym}}(z)\,\widehat{\mu}^{(k)\,\mathrm{sym}}(c^{-1}z) \tag{7.2}$$

for all $k \in \mathbb{Z}$, where $\widehat{\rho}^{(k)\,\mathrm{sym}}(z)$ and $\widehat{\mu}^{(k)\,\mathrm{sym}}(z)$ denote the characteristic functions of $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$. Thus $\mu^{(k)\,\mathrm{sym}}$ is again c^{-1}-decomposable. These symmetrizations have the following remarkable property.

Lemma 7.1. *Define $(c', p', q', r') = (c, r, q, p)$ and let $\rho'^{(k)}$ and $\mu'^{(k)}$ be the distributions corresponding to $\rho^{(k)}$ and $\mu^{(k)}$ with (c', p', q', r') used in place of (c, p, q, r). Let $\rho'^{(k)\,\mathrm{sym}}$ and $\mu'^{(k)\,\mathrm{sym}}$ be their symmetrizations. Then*

$$\rho'^{(k)\,\mathrm{sym}} = \rho^{(k)\,\mathrm{sym}}, \tag{7.3}$$

$$\mu'^{(k)\,\mathrm{sym}} = \mu^{(k)\,\mathrm{sym}} \tag{7.4}$$

for $k \in \mathbb{Z}$.

Proof. It follows from (2.4) that

$$\widehat{\rho}^{(k)\,\mathrm{sym}}(z) = \left|\frac{p + re^{ic^{-k}z}}{1 - qe^{iz}}\right|^2 = \left|\frac{pe^{-ic^{-k}z} + r}{1 - qe^{iz}}\right|^2 = \left|\frac{r + pe^{ic^{-k}z}}{1 - qe^{iz}}\right|^2.$$

Hence $\rho'^{(k)\,\mathrm{sym}}$ and $\rho^{(k)\,\mathrm{sym}}$ have an identical characteristic function, that is, (7.3) is true. Then (7.4) follows as in (2.2). □

We also use the following general result.

Lemma 7.2. *Suppose that σ is a distribution on $\mathbb{R}$.*

(i) *If $\sigma \in ID \cup ID^0$, then $\sigma^{\mathrm{sym}} \in ID \cup ID^0$.*

(ii) *If $\sigma \in ID^0$ with quasi-Lévy measure being concentrated on $(0, \infty)$, then $\sigma^{\mathrm{sym}} \in ID^0$.*

Proof. (i) It is clear that if σ satisfies (1.9) with γ, a and ν_σ, then $\sigma^{\mathrm{sym}} \in ID \cup ID^0$ satisfying (1.9) with $\gamma^{\mathrm{sym}} = 0, a^{\mathrm{sym}} = 2a$ and $\nu_{\sigma^{\mathrm{sym}}}$ given by $\nu_{\sigma^{\mathrm{sym}}}(B) = \nu_\sigma(B) + \nu_\sigma(-B)$ for $B \in \mathcal{B}(\mathbb{R})$.

(ii) If $\sigma \in ID^0$, then $a < 0$ or ν_σ has nontrivial negative part. Hence it follows from the proof of (i) that if $a < 0$, then $a^{\mathrm{sym}} < 0$, and if ν_σ has nontrivial negative part and is concentrated on $(0, \infty)$, then σ^{sym} has non-trivial negative part. In both cases it holds $\sigma^{\mathrm{sym}} \in ID^0$. □

Theorem 7.3. *Let $k \in \mathbb{Z}$.*

(i) *If $p = 0$ or if $r = 0$, then $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$ are in ID.*

(ii) *If $p \neq r$, then $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$ are in $ID \cup ID^0$.*

(iii) *If $p = r$, then $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$ are in ID^{00}.*

Proof. (i) and (ii) are clear from Theorem 4.1 (i)-(iii) and Lemma 7.2, while (iii) follows from the fact that $\widehat{\rho}^{(k)}(z)$ and hence $\widehat{\mu}^{(k)}(z)$ have zero points for $p = r$ by (2.4). □

In studying infinite divisibility properties of $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$, we will only consider whether they are infinitely divisible or not in the case where

$$p > 0, \quad r > 0, \quad \text{and} \quad p \neq r, \tag{7.5}$$

as we have Theorem 7.3.

Theorem 7.4. *Let $k \in \mathbb{Z}$ and assume (7.5). Let $\rho'^{(k)}$ and $\mu'^{(k)}$ be defined as in Lemma 7.1.*

(i) *$\rho^{(k)\,\mathrm{sym}} \in ID$ if and only if $\rho^{(k)} \in ID$ or $\rho'^{(k)} \in ID$.*

(ii) $\mu^{(k)\,\mathrm{sym}} \in ID$ *if only if* $\mu^{(k)} \in ID$ *or* $\mu'^{(k)} \in ID$.

Proof. The 'if' part of (i) follows from (7.1) and (7.3). To see the 'only if' part, suppose that $\rho^{(k)\,\mathrm{sym}} \in ID$. If $r < p$, then $\rho^{(k)} \in ID \cup ID^0$ with quasi-Lévy measure being concentrated on $(0, \infty)$ by Theorem 4.1 (ii), and $\rho^{(k)} \in ID$ from Lemma 7.2 (ii). If $r > p$, then $r' < p'$ and the same reasoning for $\rho'^{(k)}$ combined with (7.3) shows that $\rho'^{(k)} \in ID$. Hence (i) is true. We obtain (ii) in the same way. □

We can now give necessary and sufficient conditions for $\rho^{(k)\,\mathrm{sym}}$ and $\mu^{(k)\,\mathrm{sym}}$ being infinitely divisible. For $k = 0$ in (i) below, the corresponding conditions were already obtained in Theorem 2.2 of [12], but thanks to Theorem 7.4, a new and much shorter proof can now be given for that part of Theorem 2.2 in [12].

Theorem 7.5. *Let* $k \in \mathbb{Z}$ *and assume* (7.5).

(i) *Let* $k = 0$. *If* $(r/p) \wedge (p/r) \le q$, *then* $\rho^{(0)\,\mathrm{sym}}, \mu^{(0)\,\mathrm{sym}} \in ID$. *Conversely, if* $(r/p) \wedge (p/r) > q$, *then* $\rho^{(0)\,\mathrm{sym}}, \mu^{(0)\,\mathrm{sym}} \in ID^0$.

(ii) *Let* $k > 0$. *Then* $\rho^{(k)\,\mathrm{sym}} \in ID$ *if and only if* $c^k = 2$ *and* $(r/p)^2 \wedge (p/r)^2 \le q$.

(iii) *Let* $k > 0$. *Then* $\mu^{(k)\,\mathrm{sym}} \in ID$ *if and only if one of the following holds:* (a) $(r/p) \wedge (p/r) \le q$; (b) $c^l = 2$ *for some* $l \in \{1, 2, \dots, k\}$ *and* $(r/p)^2 \wedge (p/r)^2 \le q$.

(iv) *Let* $k < 0$. *Then* $\rho^{(k)\,\mathrm{sym}} \in ID$ *if and only if* $2c^{|k|} \in \mathbb{N}$ *and* $q^l \ge (l/2)[(r/p)^2 \wedge (p/r)^2]$ *for* $l = 2c^{|k|}$.

(v) *Let* $k < 0$. *If* $2c^j \notin \mathbb{N}$ *for all integers* j *satisfying* $j \ge |k|$, *then* $\mu^{(k)\,\mathrm{sym}} \in ID^0$.

(vi) *Let* $k < 0$. *Suppose that* $c^j \in \mathbb{N}$ *for some* $j \in \mathbb{N}$. *Let* l *be the smallest of such* j *and let* $\alpha = c^l$, $\beta := \lceil |k|/l \rceil$ *and* $h_{\alpha,\beta}$ *be defined by* (6.1). *Then* $\mu^{(k)\,\mathrm{sym}} \in ID$ *if and only if* $q > 0$ *and* $h_{\alpha,\beta}(q^{\alpha^\beta}) \ge (r/p) \wedge (p/r)$.

(vii) *Let* $k < 0$. *Suppose that* $2c^j \in \mathbb{N}_{\mathrm{odd}}$ *for some* $j \in \mathbb{N}$ *with* $j \ge |k|$. *Then* j *is unique. Let* $\alpha = c^j$ *and suppose that* $2\alpha \ge 5$. *Then* $\mu^{(k)\,\mathrm{sym}} \in ID$ *if and only if* $q^{2\alpha} + ((r/p) \wedge (p/r))^{2\alpha} \ge \alpha((r/p) \wedge (p/r))^2$.

Proof. All assertions are immediate consequences of Theorem 7.4, Theorem 4.1, and the corresponding results obtained earlier. For (i), use Proposition 5.1, for (ii) Theorem 5.2, for (iii) Theorem 5.3, for (iv) Theorem 6.1, and for (v) – (vii) use Theorem 6.3. □

Conditions for $\mu^{(k)\,\mathrm{sym}} \in ID$ when $2c^j = 3$ with $j, -k \in \mathbb{N}$ and $j \ge |k|$ can be written down similarly as in (vii) above with the aid of Theorem 6.3 $(\mathrm{iii})_2$.

Corollary 7.6. *For each* $k \in \mathbb{Z} \backslash \{0\}$, *parameters* c, p, q, r *exist such that* $\mu^{(k)\,\mathrm{sym}} \in ID$ *and* $\rho^{(k)\,\mathrm{sym}} \in ID^0$.

The proof is immediate from Theorem 7.5. Corollary 7.6 gives symmetric examples of infinitely divisible distributions which are b-decomposable without infinitely divisible factor, the phenomenon first observed by Niedbalska-Rajba [16].

The next corollary gives further examples of a phenomenon first observed by Gnedenko and Kolmogorov [8], p. 82. Its proof is immediate from Theorem 4.1, Proposition 5.1, Theorems 5.2, 5.3, and 7.5.

Corollary 7.7. *For each $k \in \mathbb{Z}$, there is a case where $\rho^{(k)\,\mathrm{sym}} \in ID$ with $\rho^{(k)} \in ID^0$ and there is a case where $\mu^{(k)\,\mathrm{sym}} \in ID$ with $\mu^{(k)} \in ID^0$.*

Let us give the analogues of Theorems 4.3, 6.6, and 5.5.

Theorem 7.8. *Let $k \in \mathbb{Z}$ and the parameters c, p, q, r be fixed. If $\mu^{(k)\,\mathrm{sym}} \in ID$, then $\mu^{(k+1)\,\mathrm{sym}} \in ID$.*

Proof. From (2.8) follows

$$\widehat{\mu}^{(k+1)\,\mathrm{sym}}(z) = \widehat{\mu}^{(k)\,\mathrm{sym}}(c^{-1}z) \left| \frac{1-q}{1-qe^{iz}} \right|^2,$$

and the second factor in the right-hand side is an infinitely divisible characteristic function. □

Theorem 7.9. *Let $c > 1$ and the parameters p, q, r be fixed such that $p > 0$ and $r > 0$. Then there is $k_0 \in \mathbb{Z}$ such that, for every $k \in \mathbb{Z}$ with $k < k_0$, $\mu^{(k)\,\mathrm{sym}} \notin ID$.*

Proof. For $r = p$, the assertion is obvious by Theorem 7.3. For $r \neq p$ it follows from Theorems 4.1, 6.6 and 7.4. □

Theorem 7.10. *Assume* (7.5). *Then $\mu^{(k)\,\mathrm{sym}} \in ID^0$ for all $k \in \mathbb{Z}$ if and only if one of the following holds:* (a) $(r/p)^2 \wedge (p/r)^2 > q$; (b) $(r/p) \wedge (p/r) > q$ *and* $c^m \neq 2$ *for all* $m \in \mathbb{N}$.

Proof. For fixed $k \in \mathbb{N}$, it follows from Theorem 7.5 (iii) that $\mu^{(k)\,\mathrm{sym}}$ is non-infinitely divisible if and only if one of the following holds: (a) $(r/p)^2 \wedge (p/r)^2 > q$; (b) $(r/p) \wedge (p/r) > q$ and $c^m \neq 2$ for all $m \in \{1, 2, \ldots, k\}$. Our assertion is obtained from this. □

Some continuity properties of the symmetrizations of $\mu^{(k)}$ are added.

Theorem 7.11. *Let $k \in \mathbb{Z}$ and the parameters c, p, q, r be fixed. Then:*

(i) *$\mu^{(k)\,\mathrm{sym}}$ is absolutely continuous if and only if $\mu^{(0)\,\mathrm{sym}}$ is absolutely continuous.*

(ii) $\mu^{(k)\,\mathrm{sym}}$ *is continuous-singular if and only if* $\mu^{(0)\,\mathrm{sym}}$ *is continuous-singular.*

(iii) $\dim(\mu^{(k)\,\mathrm{sym}}) = \dim(\mu^{(0)\,\mathrm{sym}})$.

(iv) $\dim(\mu^{(k)\,\mathrm{sym}}) \le H(\rho^{(k)\,\mathrm{sym}})/\log c \le 2H(\rho^{(k)})/\log c$.

Proof. It follows from (2.6) that

$$\widehat{\mu}^{(k)\,\mathrm{sym}}(z) = \widehat{\mu}^{(k+1)\,\mathrm{sym}}(z)\,|p_0 + r_0 e^{ic^{-k}z}|^2,$$

where $p_0 = p/(p+r)$ and $r_0 = r/(p+r)$. Since

$$|p_0 + r_0 e^{ic^{-k}z}|^2 = (p_0 + r_0 e^{ic^{-k}z})\,(p_0 + r_0 e^{-ic^{-k}z}) = p_0^2 + r_0^2 + p_0 r_0 (e^{ic^{-k}z} + e^{-ic^{-k}z}),$$

we have

$$\mu^{(k)\,\mathrm{sym}} = \mu^{(k+1)\,\mathrm{sym}} * [(p_0^2 + r_0^2)\delta_0 + p_0 r_0 (\delta_{c^{-k}} + \delta_{-c^{-k}})],$$

that is,

$$\mu^{(k)\,\mathrm{sym}}(B) = (p_0^2 + r_0^2)\mu^{(k+1)\,\mathrm{sym}}(B) + p_0 r_0 [\mu^{(k+1)\,\mathrm{sym}}(B - c^{-k}) + \mu^{(k+1)\,\mathrm{sym}}(B + c^{-k})]$$

for $B \in \mathcal{B}(\mathbb{R})$. Hence an argument similar to the proof of Theorem 3.1 works to show (i)–(iii), since $\mu^{(k)\,\mathrm{sym}}$ is c^{-1}-decomposable by (7.2) and hence either absolutely continuous, continuous singular, or a Dirac measure. Assertion (iv) follows from Watanabe's theorem [20] and E29.23 of [19]. □

The statement of Theorem 3.3 is true for $\mu^{(k)\,\mathrm{sym}}$ in place of $\mu^{(k)}$, except that $\log 3$ should be replaced by $2\log 3$.

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