Absolute Moments of Generalized Hyperbolic Distributions and Approximate Scaling of Normal Inverse Gaussian Lévy Processes

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ABSTRACT. Expressions for (absolute) moments of generalized hyperbolic and normal inverse Gaussian (NIG) laws are given in terms of moments of the corresponding symmetric laws. For the (absolute) moments centred at the location parameter μ explicit expressions as series containing Bessel functions are provided. Furthermore, the derivatives of the logarithms of absolute μ-centred moments with respect to the logarithm of time are calculated explicitly for NIG Lévy processes. Computer implementation of the formulae obtained is briefly discussed. Finally, some further insight into the apparent scaling behaviour of NIG Lévy processes is gained.

Key words: generalized inverse Gaussian distribution, normal inverse Gaussian distribution, scaling

1. Introduction

The generalized hyperbolic (GH) distribution was introduced in Barndorff-Nielsen (1977) in connection to a study of the grain-size distribution of wind-blown sand. Since then it has been used in many different areas. Before outlining the contents of the present paper, we now give a brief overview of various applications of the GH laws and the associated Lévy processes.

The original paper by Barndorff-Nielsen (1977) was focused on the special case of the hyperbolic law. That law and its applicability have been further discussed, *inter alia*, in Barndorff-Nielsen *et al*. (1983, 1985) (particle size distributions of sand), Barndorff-Nielsen & Christiansen (1988), Hartmann & Bowman (1993), Sutherland & Lee (1994) and references therein (coastal sediments), Xu *et al*. (1993) and their list of references (fluid sprays). In Barndorff-Nielsen (1982) the appearance of the three-dimensional hyperbolic law in relativistic statistical physics was pointed out. Other areas, where hyperbolic distributions have been employed, include biology (e.g. Blæsild, 1981) and primary magnetization of lava flows (cf. Kristjansson & McDougall, 1982). Furthermore, in Barndorff-Nielsen *et al*. (1989) the hyperbolic distribution is employed to model wind shear data of landing aircrafts parsimoniously. See also Barndorff-Nielsen (1979) for applications in turbulence.

Moreover, Barndorff-Nielsen *et al*. (2004) recently demonstrated (following an indication in Barndorff-Nielsen, 1998a) that the normal inverse Gaussian (NIG) law, another important special case of the GH law, is capable of describing velocity data from turbulence experiments with high accuracy. Eriksson *et al*. (2004) employ the NIG distribution to approximate other (unknown) probability distributions.

In recent years many authors have successfully fitted generalized hyperbolic distributions and in particular NIG laws to returns in financial time series; see Eberlein & Keller (1995), Prause (1997, 1999), Barndorff-Nielsen (1997), Barndorff-Nielsen & Shephard (2001a,b) and references therein, Schoutens (2003), Barndorff-Nielsen & Shephard (2005). Benth & Šaltytė-Benth (2005) have recently put forth a model for Norwegian temperature data driven by a GH Lévy process.

This has, in particular, led to modelling the time dynamics of financial markets by stochastic processes using generalized hyperbolic or normal inverse Gaussian laws and associated Lévy processes as building blocks (e.g. Rydberg, 1997, 1999; Bibby & Sørensen, 1997; Barndorff-Nielsen, 1998b, 2001; Prause, 1999; Raible, 2000; Barndorff-Nielsen & Shephard, 2001a, 2005 and references therein; Eberlein, 2001; Schoutens, 2003; Cont & Tankov, 2004; Rasmus *et al.* 2004; Emmer & Klüppelberg, 2004; Mencia & Sentana, 2004).

One of the reasons, why the GH distribution is used in such a variety of situations, is that it is not only flexible enough to fit many different data sets well, but also is rather tractable analytically, and many important properties (density, characteristic function, cumulant transform, Lévy measure, etc.) are known. Some of these properties are recalled in section 2. Yet, until now, no details on absolute moments of arbitrary order $r > 0$ are known, except for $r = 1$. Thus we derive, in section 3, formulae for the (absolute) moments of arbitrary order $r > 0$ of the generalized hyperbolic distribution in terms of moments of the corresponding symmetric GH law. For μ-centred (absolute) moments, i.e. moments centred at the location parameter μ, we are able to give explicit formulae using Bessel functions. From these general formulae we will then, as special cases, obtain formulae for the absolute moments of the NIG law and NIG Lévy process. We especially focus on the NIG case because of its wide applicability and tractability. In particular, the NIG Lévy process has a marginal NIG distribution at all times, an appealing feature not shared by the general GH Lévy process.

Due to Kolmogorov's famous laws for homogeneous and isotropic turbulence (see, for instance, Frisch, 1995) scaling is an important issue when considering turbulence data and models. Ongoing research indicates that the time transformation carried out in Barndorff-Nielsen *et al.* (2004) leads to a process with NIG marginals and very strong apparent scaling.

The question of the possible relevance of scaling in finance was raised by Mandelbrot (1963) and has since been discussed by a number of authors, see, in particular, Müller *et al.* (1990), Guillaume *et al.* (1997) and Mandelbrot (1997). More recently the question was taken up by Barndorff-Nielsen & Prause (2001), who showed that an NIG Lévy process may exhibit moment behaviour which is very close to scaling. However, they solely studied the first absolute moment and obtained analytic results only in the case of a symmetric NIG Lévy process. As part of this paper we generalize their findings to the skewed case and higher order moments.

Based on the explicit general formulae for μ-centred moments of the NIG law, resp. NIG Lévy process, we are able to deduce analytic results for the approximate scaling of an NIG Lévy process, namely explicit expressions for the derivative of the logarithm of the μ-centred absolute moments of the NIG Lévy process with respect to the logarithm of time.

In the final sections we discuss the numerical implementation of the formulae obtained and give numerical examples for the apparent scaling present in NIG Lévy processes.

Our results show that, in particular, the occurrence of empirical scaling laws is not bound to necessitate the use of self-similar or even multifractal processes for modelling, as this type of behaviour is already exhibited by such simple a model as an NIG Lévy process, when looking at the relevant time horizons. For a survey of the theory and (approximate) occurrence of self-similarity and scaling, see Embrechts & Maejima (2002).

2. Generalized hyperbolic and inverse Gaussian distributions

In this paper, we consider the class of one-dimensional GH distributions and the subclass of NIG distributions in particular. The GH law was, as already noted, introduced in Barndorff-Nielsen (1977) and its properties were further studied in Barndorff-Nielsen (1978a), Barndorff-Nielsen & Blæsild (1981) and Blæsild & Jensen (1981). Some recent results, in particular regarding the multivariate GH laws, can be found in Prause (1999), Eberlein (2001), Eberlein & Hammerstein (2004) and Mencia & Sentana (2004).

We denote the (one-dimensional) generalized hyperbolic distribution by GH (v, α, β, μ, δ) and characterize it via its probability density given by:

$$p(x; v, \alpha, \beta, \mu, \delta) = \frac{\bar{\gamma}^{v} \bar{\alpha}^{1/2-v}}{\sqrt{2\pi}\delta K_v(\bar{\gamma})} \left(1 + \frac{(x-\mu)^2}{\delta^2}\right)^{v/2-1/4} \cdot K_{v-1/2}\left(\bar{\alpha}\sqrt{1 + \frac{(x-\mu)^2}{\delta^2}}\right) \mathrm{e}^{\beta(x-\mu)} \tag{1}$$

for $x \in \mathbb{R}$ and where the parameters satisfy $v \in \mathbb{R}$, $0 \leq |\beta| < \alpha$, $\mu \in \mathbb{R}$, $\delta \in \mathbb{R}_{>0}$, and $\gamma := \sqrt{\alpha^2 - \beta^2}$, $\bar{\alpha} := \delta\alpha$, $\bar{\beta} := \delta\beta$, $\bar{\gamma} := \delta\gamma$. Here α can be interpreted as a shape, β as a skewness, μ as a location and δ as a scaling parameter; finally, v characterizes subclasses and primarily influences the tail behaviour.

Furthermore, $K_v(\cdot)$ denotes the modified Bessel function of the third kind and order $v \in \mathbb{R}$. For a comprehensive discussion of Bessel functions of complex arguments see Watson (1952). Jørgensen (1982) contains an appendix listing important properties of Bessel functions of the third kind and related functions. Most of these properties can also be found in standard reference books like Gradshteyn & Ryzhik (1965) or Bronstein *et al.* (2000). For the following we need to know that K_v is defined on the positive half plane $D = \{z \in \mathbb{C} : \Re(z) > 0\}$ of the complex numbers and is holomorphic on D. From Watson (1952, p. 182) or Jørgensen (1982, p. 170) we have the representation

$$K_v(z) = \frac{1}{2}\int_0^{\infty} y^{v-1}\, \mathrm{e}^{-\frac{1}{2}z(y+y^{-1})}\, \mathrm{d}y, \tag{2}$$

which shows the strict positivity of K_v on $\mathbb{R}_{>0}$. The substitution $x := y^{-1}$ immediately gives $K_{-v} = K_v$. Furthermore, $K_v(z)$ is obviously monotonically decreasing in z on $\mathbb{R}_{>0}$. From the alternative representation

$$K_v(z) = \int_0^{\infty} \mathrm{e}^{-z\cosh(t)} \cosh(vt)\, \mathrm{d}t \tag{3}$$

(cf. Watson, 1952, p. 181) one reads off that, for fixed $z \in \mathbb{R}_{>0}$, $K_v(z)$ is strictly increasing in v for $v \in \mathbb{R}_{\geq 0}$.

There are several popular subclasses contained within the GH laws. For $v = 1$ the hyperbolic and for $v = -1/2$ the normal inverse Gaussian distributions are obtained. The normal, exponential, Laplace, Variance-Gamma and Student-t distributions are among many others limiting cases of the GH distribution (cf. Eberlein & Hammerstein, 2004 for a comprehensive analysis).

Alternatively one often uses

$$\rho := \frac{\beta}{\alpha} = \frac{\bar{\beta}}{\bar{\alpha}},\ \xi := (1+\bar{\gamma})^{-1/2} \text{ and } \chi := \rho\xi$$

to parameterize the GH law, as these quantities are invariant under location-scale changes and χ, resp. ξ, can be interpreted as a skewness, resp. kurtosis, measure. Moreover, the parameter restrictions imply $0 < |\chi| < \xi < 1$. For fixed v this gives rise to the use of *shape triangles* as a graphical tool to study generalized hyperbolic distributions (see e.g. Barndorff-Nielsen *et al.*, 1983, 1985; Barndorff-Nielsen & Christiansen, 1988; Rydberg, 1997; Prause, 1999).

A very useful representation in law of the generalized hyperbolic distribution can be given using the generalized inverse Gaussian distribution. The generalized inverse Gaussian distribution GIG(ν, δ, γ) with parameters $\nu \in \mathbb{R}$, γ, $\delta \in \mathbb{R}_{\geq 0}$ and $\gamma + \delta > 0$ is the distribution on $\mathbb{R}_{>0}$ which has probability density function

$$\begin{aligned} p(x;\nu,\delta,\gamma) &= \frac{(\gamma/\delta)^{\nu}}{2K_{\nu}(\delta\gamma)}x^{\nu-1}\exp\left(-\frac{1}{2}(\delta^2x^{-1}+\gamma^2x)\right) \\ &= \frac{\bar{\gamma}^{\nu}}{2K_{\nu}(\bar{\gamma})}\delta^{-2\nu}x^{\nu-1}\exp\left(-\frac{1}{2}(\delta^2x^{-1}+\bar{\gamma}^2\delta^{-2}x)\right). \end{aligned} \tag{4}$$

For more information on the GIG law we refer to Jørgensen (1982) and for an interpretation in terms of hitting times to Barndorff-Nielsen *et al.* (1978). The following normal variance–mean mixture representation of the generalized hyperbolic law holds.

Lemma 1

Let $X \sim$ GH(ν, α, β, μ, δ), $V \sim$ GIG(ν, δ, γ) *with* $\gamma = \sqrt{\alpha^2 - \beta^2}$ *and* $\varepsilon \sim N(0, 1)$, *where* V *and* ε *are independent*, *then:*

$$X \stackrel{D}{=} \mu + \beta V + \sqrt{V}\varepsilon.$$

(For a general overview over normal variance–mean mixtures see Barndorff-Nielsen *et al.*, 1982.)

Furthermore, the cumulant function of the generalized hyperbolic law $X \sim$ GH(ν, α, β, μ, δ) is given by

$$K(\theta \ddagger X) = \frac{\nu}{2}\log\left(\frac{\gamma}{\alpha^2-(\beta+\theta)^2}\right) + \log\left(\frac{K_{\nu}\left(\delta\sqrt{\alpha^2-(\beta+\theta)^2}\right)}{K_{\nu}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right) + \theta\mu. \tag{5}$$

Obviously $K(\theta \ddagger X)$ is defined for all $\theta \in \mathbb{R}$ with $|\beta + \theta| < \alpha$. From this fact and Barndorff-Nielsen (1978b, Corollary 7.1) we immediately obtain:

Lemma 2

Assume $X \sim$ GH(ν, α, β, μ, δ). *Then* $X \in L^p$ *for all* $p > 0$, *i.e.* $E(|X|^p)$ *exists for all* $p > 0$.

One immediately calculates the expected value and variance of a GH-distributed random variate X to be

$$E(X) = \mu + \beta\frac{\delta K_{\nu+1}(\bar{\gamma})}{\gamma K_{\nu}(\bar{\gamma})},$$

$$\mathrm{Var}(X) = \delta^2\left(\frac{K_{\nu+1}(\bar{\gamma})}{\bar{\gamma}K_{\nu}(\bar{\gamma})} + \frac{\beta^2}{\gamma^2}\left(\frac{K_{\nu+2}(\bar{\gamma})}{K_{\nu}(\bar{\gamma})} - \left(\frac{K_{\nu+1}(\bar{\gamma})}{K_{\nu}(\bar{\gamma})}\right)^2\right)\right).$$

Higher order cumulants can also be calculated, but the expressions become more and more complicated.

In the GH law the existence of moments of all orders is combined with semi-heavy tails

$$p(x;\nu,\alpha,\beta,0,\delta) \sim C|x|^{\nu-1}\exp((\mp\alpha+\beta)x) \text{ as } x \to \pm\infty \tag{6}$$

for some constant C.

In Fig. 1 the densities of NIG distributions fitted to turbulent velocity increments (from data set I of Barndorff-Nielsen *et al*., 2004; with a detailed description of the data) at different lags are plotted on a logarithmic scale together with histograms of the original data. The graphs were obtained using the programme 'hyp' (Blæsild & Sørensen, 1992). They exemplify the rich variety of distributional shapes one can already get from the NIG law. Letting ν vary offers the possibility to get even more different shapes (see e.g. Eberlein & Özkan, 2003 for some hyperbolic fits). Note especially the marked differences in the centre of the distributions and the difference in how the asymptote given by (6) is approached. For a lag of 9000 the shape is already very close to the quadratic one of the Gaussian law.

Moreover, the GH law is infinitely divisible (in fact, self-decomposable) and leads thus to an associated Lévy process. Yet, the GH distribution is not closed under convolution, but the NIG distribution has this property, so that all marginal distributions of a Lévy process associated with an NIG distribution belong to the NIG class.

The above facts and the analytical tractability due to the existence of explicit expressions for the density, the cumulant function and related functions makes using the GH law appealing in many different areas, as already pointed out in the Introduction.

3. Moments and absolute moments of GH laws

In this section, we give expressions for different (absolute) moments of arbitrary GH distributions in terms of moments of corresponding symmetric GH distributions. Based upon this

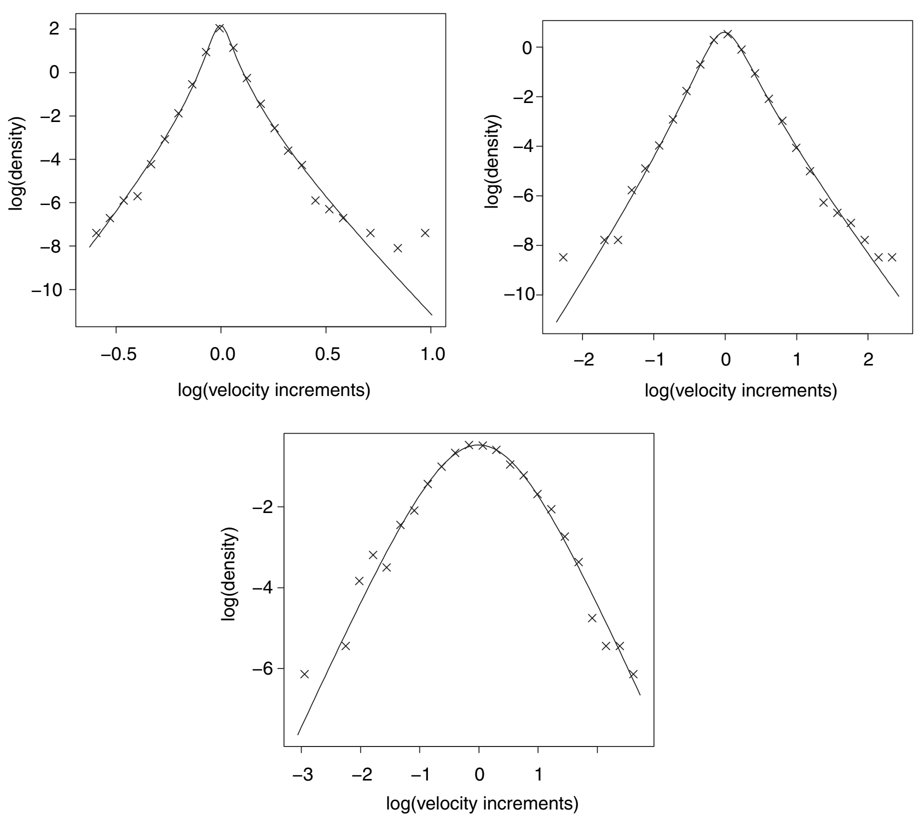

Fig. 1. Log-density (solid line) of NIG distribution fitted to velocity increments at lags 12 (upper left), 500 (upper right) and 9000 (lower centre) and log-histogram (only top end points are given (×)).

we obtain explicit expressions for μ-centred (absolute) moments of GH distributions, employing the variance–mean mixture representation.

Theorem 1

Let $X \sim \mathrm{GH}(\nu, \alpha, \beta, \mu, \delta)$, $Y \sim \mathrm{GH}(\nu, \alpha, 0, \mu, \delta)$, *then for every* $r > 0$ *and* $n \in \mathbb{N}$*:*

$$(i)\quad E(X^n) = \left(\frac{\bar\gamma}{\bar\alpha}\right)^{\nu} \frac{K_\nu(\bar\alpha)}{K_\nu(\bar\gamma)} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} E(Y^n (Y-\mu)^k)$$

$$(ii)\quad E(|X|^r) = \left(\frac{\bar\gamma}{\bar\alpha}\right)^{\nu} \frac{K_\nu(\bar\alpha)}{K_\nu(\bar\gamma)} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} E(|Y|^r (Y-\mu)^k)$$

$$(iii)\quad E((X-\mu)^n) = \left(\frac{\bar\gamma}{\bar\alpha}\right)^{\nu} \frac{K_\nu(\bar\alpha)}{K_\nu(\bar\gamma)} \sum_{k=0}^{\infty} \frac{\beta^{2k+m}}{(2k+m)!} E((Y-\mu)^{2k+m+n})$$

$$(iv)\quad E(|X-\mu|^r) = \left(\frac{\bar\gamma}{\bar\alpha}\right)^{\nu} \frac{K_\nu(\bar\alpha)}{K_\nu(\bar\gamma)} \sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} E(|Y-\mu|^{2k+r}),$$

where $m := n \bmod 2$. *All moments above are finite.*

Note that from the cumulant function we have $E(Y) = \mu$ and $E(X) = \mu + \beta\delta K_{\nu+1}(\bar\gamma)/(\gamma K_\nu(\bar\gamma))$, as stated before. Hence, we have that $E((Y-\mu)^r)$ are central moments, whereas $E((X-\mu)^r)$ are in general just μ-centred moments. Note also that sgn $E((X-\mu)^n =$ sgn β for all odd n.

Proof. We will only prove (ii), as the proofs of the other formulae proceed along the same lines, except that to obtain (iii) and (iv) one notes in the final step that odd central moments of Y vanish, as the distribution of Y is symmetric around μ.

The series representation of the exponential function gives

$$\begin{aligned} E(|X|^r) &= \int_{\mathbb{R}} \frac{\bar\gamma^\nu \bar\alpha^{1/2-\nu}}{\sqrt{2\pi}\delta K_\nu(\bar\gamma)} \left(1 + \frac{(x-\mu)^2}{\delta^2}\right)^{\nu/2-1/4} \\ &\quad \cdot K_{\nu-1/2}\left(\bar\alpha\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right) e^{\beta(x-\mu)} |x|^r \, dx \\ &= \int_{\mathbb{R}} \sum_{k=0}^{\infty} \frac{\bar\gamma^\nu \bar\alpha^{1/2-\nu}}{\sqrt{2\pi}\delta K_\nu(\bar\gamma)} \left(1 + \frac{(x-\mu)^2}{\delta^2}\right)^{\nu/2-1/4} \\ &\quad \cdot K_{\nu-1/2}\left(\bar\alpha\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right) \frac{\beta^k}{k!} (x-\mu)^k |x|^r \, dx. \end{aligned}$$

The integrals exist (cf. lemma 2) and the same is true with β changed to $-\beta$. This implies that the integrals

$$\int_{\mu}^{\infty} \frac{\bar\gamma^\nu \bar\alpha^{1/2-\nu}}{\sqrt{2\pi}\delta K_\nu(\bar\gamma)} \left(1 + \frac{(x-\mu)^2}{\delta^2}\right)^{\nu/2-1/4} K_{\nu-1/2}\left(\bar\alpha\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right) e^{|\beta(x-\mu)|} |x|^r \, dx$$

and

$$\int_{-\infty}^{\mu} \frac{\bar\gamma^\nu \bar\alpha^{1/2-\nu}}{\sqrt{2\pi}\delta K_\nu(\bar\gamma)} \left(1 + \frac{(x-\mu)^2}{\delta^2}\right)^{\nu/2-1/4} K_{\nu-1/2}\left(\bar\alpha\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right) e^{|\beta(x-\mu)|} |x|^r \, dx$$

and hence the integral

$$\int_{\mathbb{R}} \frac{\bar{\gamma}^{\nu}\bar{\alpha}^{1/2-\nu}}{\sqrt{2\pi}\delta K_{\nu}(\bar{\gamma})}\left(1+\frac{(x-\mu)^2}{\delta^2}\right)^{\nu/2-1/4} K_{\nu-1/2}\left(\bar{\alpha}\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right)\mathrm{e}^{|\beta(x-\mu)|}|x|^r\,\mathrm{d}x$$

exist. Using the last one as majorant, Lebesgue's convergence theorem gives

$$\begin{aligned}E(|X|^r) &= \sum_{k=0}^{\infty}\int_{\mathbb{R}} \frac{\bar{\gamma}^{\nu}\bar{\alpha}^{1/2-\nu}}{\sqrt{2\pi}\delta K_{\nu}(\bar{\gamma})}\left(1+\frac{(x-\mu)^2}{\delta^2}\right)^{\nu/2-1/4}\\ &\quad\cdot K_{\nu-1/2}\left(\bar{\alpha}\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right)\frac{\beta^k}{k!}(x-\mu)^k|x|^r\,\mathrm{d}x\\ &= \sum_{k=0}^{\infty}\frac{\beta^k}{k!}\left(\frac{\bar{\gamma}}{\bar{\alpha}}\right)^{\nu}\frac{K_{\nu}(\bar{\alpha})}{K_{\nu}(\bar{\gamma})}\int_{\mathbb{R}} \frac{\bar{\alpha}^{\nu}\bar{\alpha}^{1/2-\nu}}{\sqrt{2\pi}\delta K_{\nu}(\bar{\alpha})}\left(1+\frac{(x-\mu)^2}{\delta^2}\right)^{\nu/2-1/4}\\ &\quad\cdot K_{\nu-1/2}\left(\bar{\alpha}\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right)(x-\mu)^k|x|^r\,\mathrm{d}x.\end{aligned}$$

From this we immediately conclude

$$E(|X|^r) = \left(\frac{\bar{\gamma}}{\bar{\alpha}}\right)^{\nu}\frac{K_{\nu}(\bar{\alpha})}{K_{\nu}(\bar{\gamma})}\sum_{k=0}^{\infty}\frac{\beta^k}{k!}E\left(|Y|^r(Y-\mu)^k\right).$$

Corollary 1

Let $X \sim \mathrm{GH}(\nu, \alpha, \beta, \mu, \delta)$, $V \sim \mathrm{GIG}(\nu, \delta, \alpha)$ *and* $\varepsilon \sim N(0, 1)$ *with* V *and* ε *independent, then for every* $r > 0$ *and* $n \in \mathbb{N}$:

$$(i)\quad E((X-\mu)^n) = \left(\frac{\bar{\gamma}}{\bar{\alpha}}\right)^{\nu}\frac{K_{\nu}(\bar{\alpha})}{K_{\nu}(\bar{\gamma})}\sum_{k=0}^{\infty}\frac{\beta^{2k+m}}{(2k+m)!}E\left(V^{k+(m+n)/2}\right)E\left(\epsilon^{2k+m+n}\right)$$

$$(ii)\quad E(|X-\mu|^r) = \left(\frac{\bar{\gamma}}{\bar{\alpha}}\right)^{\nu}\frac{K_{\nu}(\bar{\alpha})}{K_{\nu}(\bar{\gamma})}\sum_{k=0}^{\infty}\frac{\beta^{2k}}{(2k)!}E\left(V^{k+r/2}\right)E\left(|\epsilon|^{2k+r}\right),$$

where $m := n \bmod 2$.

Proof. Combine theorem 1 with lemma 1.

Note that we obtain the (absolute) moments of X provided $\mu = 0$ and the (absolute) central moments if $\beta = 0$. For $\beta = 0$ the above series are in fact just a single term or vanish completely.

Using the explicit expressions for the moments of GIG and normal laws, given in appendix A, it is now straightforward to obtain more explicit expressions for the μ-centred (absolute) moments of GH laws.

Theorem 2

Let $X \sim \mathrm{GH}(\nu, \alpha, \beta, \mu, \delta)$, *then for every* $r > 0$ *and* $n \in \mathbb{N}$:

$$(i)\quad E((X-\mu)^n) = \frac{2^{\lceil \frac{n}{2}\rceil}\bar{\gamma}^{\nu}\delta^{2\lceil \frac{n}{2}\rceil}\beta^m}{\sqrt{\pi}K_{\nu}(\bar{\gamma})\bar{\alpha}^{\nu+\lceil \frac{n}{2}\rceil}}\sum_{k=0}^{\infty}\frac{2^k\bar{\beta}^{2k}\Gamma\left(k+\lceil\frac{n}{2}\rceil+\frac{1}{2}\right)}{\bar{\alpha}^k(2k+m)!}K_{\nu+k+\lceil \frac{n}{2}\rceil}(\bar{\alpha})$$

$$(ii)\quad E(|X-\mu|^r) = \frac{2^{\frac{r}{2}}\bar{\gamma}^{\nu}\delta^r}{\sqrt{\pi}K_{\nu}(\bar{\gamma})\bar{\alpha}^{\nu+\frac{r}{2}}}\sum_{k=0}^{\infty}\frac{2^k\bar{\beta}^{2k}\Gamma\left(k+\frac{r}{2}+\frac{1}{2}\right)}{\bar{\alpha}^k(2k)!}K_{\nu+k+\frac{r}{2}}(\bar{\alpha}),$$

where $m := n \bmod 2$.

Proof. Combine corollary 1 with lemmas 4 and 5 noting that $(n+m) \bmod 2 = 0$ and $(m+n)/2 = (n \bmod 2 + n)/2 = \lceil n/2 \rceil$.

The absolute convergence of the series on the right-hand sides is obviously implied by the finiteness of $E((X-\mu)^n)$, resp. $E(|X-\mu|^r)$, and the positivity of all terms involved. Yet, one can also immediately give an analytic argument, which adds further insight into the convergence behaviour and is useful when one implements the above formulae on a computer (see section 6). Let

$$a_k := \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{\bar{\alpha}^k (2k)!} K_{v+k+\frac{r}{2}}(\bar{\alpha}).$$

From $K_v(z) \sim \sqrt{(\pi/2)} 2^v v^{v-1/2} \, \mathrm{e}^{-v} z^{-v}$ for $v \to \infty$ (Ismail, 1977; Jørgensen, 1982, p. 171) we obtain

$$\frac{a_{k+1}}{a_k} \sim \frac{4\bar{\beta}^2 \left(k + \frac{r}{2} + \frac{1}{2}\right)\left(k + v + \frac{r}{2}\right)\left(1 + \frac{1}{k+v+\frac{r}{2}}\right)^{k+v+\frac{r+1}{2}}}{\bar{\alpha}^2 e (2k+2)(2k+1)} \overset{k\to\infty}{\to} \left(\frac{\bar{\beta}}{\bar{\alpha}}\right)^2 < 1 \tag{7}$$

and thus the quotient criterion from standard analysis implies absolute convergence. Lemma 6, which we give in appendix B2, and its proof add some further insight into the behaviour of the series.

As a side result of theorem 2 we also obtain two identities for modified Bessel functions of the third kind.

Corollary 2

Let $x, y, z \in \mathbb{R}_{>0}$ s.t. $z = \sqrt{x^2 - y^2}$ and $v \in \mathbb{R}$ then

$$(i) \quad K_v(z) = \frac{z^v}{x^v} \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} \frac{y^{2k}}{x^k} K_{v+k}(x)$$

$$(ii) \quad z K_v(z) + y^2 K_{v+1}(z) = \frac{z^{v+1}}{x^v} \sum_{k=0}^{\infty} \frac{2k+1}{2^k \cdot k!} \frac{y^{2k}}{x^k} K_{v+k}(x).$$

Proof. Combine theorem 2 with

$$E(X) = \frac{\delta \bar{\beta}}{\bar{\gamma}} \frac{K_{v+1}(\bar{\gamma})}{K_v(\bar{\gamma})} \text{ and } E(X^2) = \delta^2 \left(\frac{K_{v+1}(\bar{\gamma})}{\bar{\gamma} K_v(\bar{\gamma})} + \frac{\bar{\beta}^2}{\bar{\gamma}^2} \frac{K_{v+2}(\bar{\gamma})}{K_v(\bar{\gamma})}\right)$$

for $X \sim \mathrm{GH}(v, \alpha, \beta, 0, \delta)$ and use $\Gamma(n + 1/2) = (2n)! \sqrt{\pi} / (2^{2n} \cdot n!)$. Finally identify x, y, z, v with $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, v + 1$.

4. Moments of NIG laws

We now turn to the normal inverse Gaussian subclass of the generalized hyperbolic law. For an overview see especially Barndorff-Nielsen (1998b). Recall that the $\mathrm{NIG}(\alpha, \beta, \mu, \delta)$ law with $0 \le |\beta| < \alpha$, $\mu \in \mathbb{R}$ and $\delta \in \mathbb{R}_{>0}$ is the special case of the $\mathrm{GH}(v, \alpha, \beta, \mu, \delta)$ law given by $v = -1/2$, as already mentioned when summarizing the properties of the GH law previously. Hence, our above calculations for (μ-centred) moments immediately lead to the following.

Corollary 3

Let $X \sim \mathrm{NIG}(\alpha, \beta, \mu, \delta)$, $Y \sim \mathrm{NIG}(\alpha, 0, \mu, \delta)$, then for every $r > 0$ and $n \in \mathbb{N}$:

$$(i)\quad E(X^n) = \mathrm{e}^{\bar{\gamma}-\bar{\alpha}} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} E(Y^n (Y-\mu)^k)$$

$$(ii)\quad E(|X|^r) = \mathrm{e}^{\bar{\gamma}-\bar{\alpha}} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} E(|Y|^r (Y-\mu)^k)$$

$$(iii)\quad E((X-\mu)^n) = \frac{2^{\lceil \frac{n}{2} \rceil + \frac{1}{2}} \delta^{2\lceil \frac{n}{2} \rceil} \beta^m}{\pi \bar{\alpha}^{\lceil \frac{n}{2} \rceil - \frac{1}{2}}} \mathrm{e}^{\bar{\gamma}} \sum_{k=0}^{\infty} \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \lceil \frac{n}{2} \rceil + \frac{1}{2}\right)}{\bar{\alpha}^k (2k+m)!} K_{k+\lceil \frac{n}{2} \rceil - \frac{1}{2}}(\bar{\alpha})$$

$$(iv)\quad E(|X-\mu|^r) = \frac{2^{\frac{r+1}{2}} \delta^r}{\pi \bar{\alpha}^{\frac{r-1}{2}}} \mathrm{e}^{\bar{\gamma}} \sum_{k=0}^{\infty} \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \frac{r+1}{2}\right)}{\bar{\alpha}^k (2k)!} K_{k+\frac{r-1}{2}}(\bar{\alpha}),$$

where $m := n \bmod 2$. *All moments above are finite.*

Proof. Follows immediately from theorems 1 and 2 using $K_{1/2}(z) = K_{-1/2}(z) = \sqrt{(\pi/2)} z^{-1/2} \mathrm{e}^{-z}$ (see e.g. Jørgensen, 1982, p. 170).

Formulae (iii) and (iv) for r equal to an even natural number can be given more explicitly using

$$K_{n+\frac{1}{2}}(z) = K_{\frac{1}{2}}(z) \left(1 + \sum_{i=1}^{n} \frac{(n+i)!}{i!(n-i)!} 2^{-i} z^{-i} \right) \tag{8}$$

for all $n \in \mathbb{N}$ (see e.g. Jørgensen, 1982, p. 170). But in order to avoid making the above formulae even more complex, we omit this.

5. Moments of NIG Lévy processes and their time-wise behaviour

Based on the above results our aim now is to generalize the findings of Barndorff-Nielsen & Prause (2001) regarding the time-wise approximate scaling behaviour of NIG Lévy processes.

5.1. Moments of NIG Lévy processes

Let $Z(t)$, $t \in \mathbb{R}_{>0}$, be the NIG(α, β, μ, δ) Lévy process, i.e. the Lévy process for which $Z(1) \sim$ NIG(α, β, μ, δ). Owing to the closedness under convolution of the NIG law, the marginal distribution of the NIG Lévy process at an arbitrary time $t \in \mathbb{R}_{>0}$ is given by NIG(α, β, $t\mu$, $t\delta$). For more background on NIG Lévy processes see in particular Barndorff-Nielsen (1998b). From our previous results we can immediately infer the following:

Corollary 4

Let $Z(t)$, $t \in \mathbb{R}_{>0}$, *be an* NIG(α, β, μ, δ) *Lévy process, then for every* $r > 0$ *and* $n \in \mathbb{N}$*:*

$$(i)\quad E((Z(t)-\mu t)^n) = \frac{2^{\lceil \frac{n}{2} \rceil + \frac{1}{2}} \delta^{2\lceil \frac{n}{2} \rceil} \beta^m}{\pi \bar{\alpha}^{\lceil \frac{n}{2} \rceil - \frac{1}{2}}} \mathrm{e}^{t\bar{\gamma}} \sum_{k=0}^{\infty} \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \lceil \frac{n}{2} \rceil + \frac{1}{2}\right)}{\bar{\alpha}^k (2k+m)!} t^{k+\lceil \frac{n}{2} \rceil + \frac{1}{2}} \cdot K_{k+\lceil \frac{n}{2} \rceil - \frac{1}{2}}(t\bar{\alpha})$$

$$(ii)\quad E(|Z(t)-\mu t|^r) = \frac{2^{\frac{r+1}{2}} \delta^r}{\pi \bar{\alpha}^{\frac{r-1}{2}}} \mathrm{e}^{t\bar{\gamma}} \sum_{k=0}^{\infty} \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \frac{r+1}{2}\right)}{\bar{\alpha}^k (2k)!} t^{k+(r+1)/2} K_{k+\frac{r-1}{2}}(t\bar{\alpha})$$

where $m := n \bmod 2$.

In appendix B2 it is shown that the moments above are analytic functions of time (lemma 6). This fact is later needed to calculate derivatives of log moments.

5.2. *Scaling and apparent scaling*

Before we now turn to discussing the scaling properties of an NIG Lévy process, let us briefly state what scaling precisely means. Let $X(t)$ be some stochastic process. We say some moment of X obeys a *scaling law*, if the logarithm of this moment is an affine function of log time, i.e. for the rth absolute moment, $\ln E(|X(t)|^r) = s_r \ln t + c_r$ for some constants $s_r, c_r \in \mathbb{R}$. Here s_r is called the *scaling coefficient*. If all (absolute) moments of X, or at least those one is interested in, follow a scaling law, we say that the process itself obeys one. For example, in the case of Brownian motion $X(t)$ with drift μ we know from $X(t) - \mu t \stackrel{D}{=} \sqrt{t}(X(1) - \mu)$ that $\ln E(|X(t) - \mu t|^r) = (r/2) \ln t + \text{constant}$ for all $r > 0$, i.e. all absolute moments exhibit scaling. More generally all self-similar processes, e.g. the strictly α-stable Lévy processes (cf. Samorodnitsky & Taqqu, 1994, Chapter 7; Sato, 1999, Chapter 3), obey a scaling law. When looking only at small changes in time the *local scaling* behaviour is determined by $d \ln E(|X(t)|^r)/d \ln t$ (in the case of the rth absolute moment). In the presence of scaling the latter derivative is constant and equals the value of the scaling coefficient. Provided some log moment of a process $X(t)$ exhibits a very close to affine dependence on log time over some time horizon of interest, we speak of *approximate* or *apparent scaling*. This is equivalent to the local scaling varying only little over the time spans considered. When working with real empirical data, it is often not possible to distinguish between apparent and strict scaling due to the randomness of the available observations. Hence, it is of interest, from a statistical point of view, whether some given theoretical process shows approximate scaling.

5.3. *The time-wise behaviour of μ-centred moments*

Let us now examine the scaling behaviour exhibited by the NIG$(\alpha, \beta, \mu, \delta)$ Lévy process $Z(t)$. For the following discussion of the time dependence of $E(|Z(t) - \mu t|^r)$ we will abbreviate the time-independent terms:

$$c(r) := \frac{2^{\frac{r+1}{2}}\delta^r}{\pi\bar{\alpha}^{\frac{r-1}{2}}} \tag{9}$$

$$a_k(r) := \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \frac{r+1}{2}\right)}{\bar{\alpha}^k (2k)!}. \tag{10}$$

If we define

$$\psi(t) := \exp(t\bar{\gamma}) \sum_{k=0}^{\infty} a_k(r) t^{k+(r+1)/2} K_{k+(r-1)/2}(t\bar{\alpha}) \tag{11}$$

and

$$\phi(t) := \ln \psi(e^t), \tag{12}$$

we have from corollary 4 that

$$E(|Z(t) - \mu t|^r) = c(r) \cdot \psi(t) \tag{13}$$

and

$$\ln E(|Z(t) - \mu t|^r) = \ln c(r) + \phi(\ln t). \tag{14}$$

Thus:

$$\frac{d\ \ln E(|Z(t)-\mu t|^r)}{d\ \ln t} = \phi'(\ln t). \tag{15}$$

Lemma 3

Let $\phi : \mathbb{R}_{>0} \to \mathbb{R}$ *be defined by* (12), *then*

$$\phi'(t) = 1 + \bar{\gamma}\ \mathrm{e}^t - \bar{\alpha}\ \mathrm{e}^t \frac{\sum_{k=0}^{\infty} a_k(r)\ \mathrm{e}^{tk} K_{k+(r-3)/2}(\mathrm{e}^t\bar{\alpha})}{\sum_{k=0}^{\infty} a_k(r)\ \mathrm{e}^{tk} K_{k+(r-1)/2}(\mathrm{e}^t\bar{\alpha})}. \tag{16}$$

Proof. Using

$$K'_v(z) = -K_{v-1}(z) - vz^{-1}K_v(z) \tag{17}$$

(see e.g. Jørgensen, 1982, p. 170 or Bronstein *et al*., 2000, p. 528), we obtain for $\psi(t)$ as defined in equation (11):

$$\begin{aligned}\psi'(t) = \exp(t\bar{\gamma})\Bigg(&\bar{\gamma}\sum_{k=0}^{\infty} a_k(r) t^{k+(r+1)/2} K_{k+(r-1)/2}(t\bar{\alpha}) + \sum_{k=0}^{\infty} a_k(r)\left(k+\frac{r+1}{2}\right)\\ &\cdot t^{k+(r-1)/2}K_{k+(r-1)/2}(t\bar{\alpha}) - \sum_{k=0}^{\infty} a_k(r) t^{k+(r+1)/2}\bar{\alpha}\\ &\cdot\left(K_{k+(r-3)/2}(t\bar{\alpha}) + \left(k+\frac{r-1}{2}\right)(t\bar{\alpha})^{-1}K_{k+\frac{r-1}{2}}(t\bar{\alpha})\right)\Bigg)\\ = \bar{\gamma}\psi(t) + t^{-1}\psi(t) - \bar{\alpha}\,\mathrm{e}^{\bar{\gamma}t}&\sum_{k=0}^{\infty} a_k(r) t^{k+(r+1)/2} K_{k+(r-3)/2}(t\bar{\alpha}).\end{aligned}$$

That we may interchange differentiation and summation above is an immediate consequence of lemma 6 and Weierstraß's theorem for sequences of holomorphic functions (see appendix B1). Hence, we get from (12)

$$\phi'(t) = \frac{\mathrm{e}^t\psi'(\mathrm{e}^t)}{\psi(\mathrm{e}^t)} = 1 + \bar{\gamma}\ \mathrm{e}^t - \bar{\alpha}\ \mathrm{e}^t \frac{\sum_{k=0}^{\infty} a_k(r)\ \mathrm{e}^{t(k+(r+1)/2)} K_{k+(r-3)/2}(\mathrm{e}^t\bar{\alpha})}{\sum_{k=0}^{\infty} a_k(r)\ \mathrm{e}^{t(k+(r+1)/2)} K_{k+(r-1)/2}(\mathrm{e}^t\bar{\alpha})}.$$

Now we can formulate our main result on the scaling behaviour of NIG Lévy processes.

Theorem 3

Let $Z(t)$, $t \in \mathbb{R}_{>0}$, *be an* NIG(α, β, μ, δ) *Lévy process*, *then*

$$\frac{d\ \ln E(|Z(t)-\mu t|^r)}{d\ \ln t} = 1 + \bar{\gamma}t - \bar{\alpha}t\frac{\sum_{k=0}^{\infty} a_k(r)t^k K_{k+(r-3)/2}(\bar{\alpha}t)}{\sum_{k=0}^{\infty} a_k(r)t^k K_{k+(r-1)/2}(\bar{\alpha}t)}$$

for every $r > 0$.

Proof. The result follows by combining lemma 3 and (15).

When comparing the above results with Barndorff-Nielsen & Prause (2001) note that they looked at the derivatives with respect to $\ln(\bar{\alpha}t)$, whereas we look at the derivative with respect to $\ln t$. The difference is related to the fact that Barndorff-Nielsen & Prause (2001) only consider the case $\beta = 0$. In the general case the parameters $\bar{\alpha}$ and $\bar{\beta}$ of the marginals at time t are both scaled with t. Hence, it is most natural and convenient to consider the change of the log moments versus the change of log time directly.

The expression for the local scaling behaviour derived in theorem 3 is in general not constant in time, hence, the absolute μ-centred moments of an NIG Lévy process do not obey a strict scaling law. Later we shall see from numerical examples that apparent scaling is common. If we look at the symmetric NIG Lévy process, i.e. $\beta = 0$, the above formula becomes

$$\frac{d \ln E(|Z(t) - \mu t|^r)}{d \ln t} = 1 + \bar{\alpha}t - \bar{\alpha}t \frac{K_{(r-3)/2}(\bar{\alpha}t)}{K_{(r-1)/2}(\bar{\alpha}t)}. \tag{18}$$

From this one deducts using $K_v = K_{-v}$ that the second μ-centred moment obeys a scaling law with slope 1, which is the same as for Brownian motion.

The aggregational Gaussianity of NIG Lévy processes (due to the central limit theorem the marginal distribution at time t of any Lévy process with finite second moment gets more and more Gaussian as t increases) becomes visible in the asymptotic scaling of the symmetric case for large times. For $r = 1$ it was already noted in Barndorff-Nielsen & Prause (2001) that the local scaling approaches 1/2 for $t \to \infty$ and hence for large t the first absolute μ-centred moment of the process seems to scale like Brownian motion. Using formula (*ii*) in corollary 4, which for $\beta = 0$ becomes

$$E(|Z(t) - \mu t|^r) = \frac{2^{\frac{r+1}{2}}\delta^r}{\pi\bar{\alpha}^{\frac{r-1}{2}}} \exp(t\bar{\alpha})\Gamma\left(\frac{r+1}{2}\right) t^{(r+1)/2} K_{\frac{r-1}{2}}(t\bar{\alpha}), \tag{19}$$

and $K_v(x) \sim \sqrt{(\pi/2)}x^{-1/2}\,\mathrm{e}^{-x}$ for $x \to \infty$ (cf. Jørgensen, 1982, p. 171 or Bronstein *et al.*, 2000) we get that $\ln E(|Z(t) - \mu t|^r) \sim (r/2) \ln t + c$ for $t \to \infty$, where $c \in \mathbb{R}$ is a constant. Note that, as usual, '$\sim$' denotes asymptotic equivalence. Hence, $E(|Z(t) - \mu t|^r)$ obeys a scaling law with slope $r/2$ for $t \to \infty$, i.e. the symmetric NIG Lévy process approaches the exact scaling behaviour of Brownian motion. This result can also be easily deduced from (18) using an asymptotic expansion of $K_{(r-3)/2}(z)/K_{(r-1)/2}(z)$ for $z \to +\infty$ (see e.g. Jørgensen, 1982, p. 173).

Studying the limiting behaviour of the absolute μ-centred moments analytically for $\beta \neq 0$ seems hardly possible. Yet, numerical studies indicate that a skewed NIG Lévy process does not scale like Brownian motion for large times in general. For example, when computing $d \ln E((Z(t))^2)/d \ln t$ of the NIG(100, 30, 0, 0.001) Lévy process for times from 1/2 to 1024 the values increase monotonically from 1.004 to 6.268. If we are, however, close to the symmetric case, i.e. if $|\beta|/\alpha$ is small, then basically the same approximate scaling behaviour is obtained as in the symmetric case. This can, in particular, be seen in the numerical data presented in section 7, where for large times the value of $d \ln E(|Z(t) - \mu t|^r)/d \ln t$ is very close to the Brownian motion scaling slope of $r/2$.

To see from (19) what happens in the symmetric case for $t \searrow 0$ we employ the fact that

$$K_v(x) \sim \begin{cases} \Gamma(v)2^{v-1}x^{-v} & \text{for } v > 0, x \searrow 0 \\ -\ln x & \text{for } v = 0, x \searrow 0 \end{cases} \tag{20}$$

(see e.g. Jørgensen, 1982, p. 171). For $r = 1$ and $t \searrow 0$ we obtain that $\ln E(|Z(t) - \mu t|)$ becomes $\ln t + \bar{\alpha}\,\mathrm{e}^{\ln t} + \ln(-\ln(t\bar{\alpha})) + c$ with $c \in \mathbb{R}$ being a constant. From this we conclude that for small values of t the first absolute μ-centred moment approximately scales with slope 1, as already noted in Barndorff-Nielsen & Prause (2001). The same asymptotic scaling slope of one holds for $r > 1$, since $\ln E(|Z(t) - \mu t|^r) \sim \ln t + \bar{\alpha}\,\mathrm{e}^{\ln t} + c(r)$ for $t \searrow 0$. Yet, a different result is obtained for $0 < r < 1$. In this case one obtains again using (20) and the identity $K_{-\nu} = K_\nu$ that $\ln E(|Z(t) - \mu t|^r) \sim r \ln t + \bar{\alpha} e^{\ln t} + c(r)$ and so there is asymptotic scaling with slope r.

6. Notes on the numerical implementation

We will now briefly discuss some issues related to the implementation of formula (ii) in corollary 4 and theorem 3 on a computer. Similar results hold for formula (i) of corollary 4. First note that (ii) in corollary 4 can be reexpressed using (10) as:

$$E(|Z(t) - \mu t|^r) = \left(\frac{2\delta^2 t}{\bar{\alpha}}\right)^{r/2} \frac{\sqrt{2t\bar{\alpha}}}{\pi} \exp(t\bar{\gamma}) \sum_{k=0}^{\infty} a_k(r) t^k K_{k+\frac{r-1}{2}}(t\bar{\alpha}). \tag{21}$$

The value of the infinite series can only be approximated. Yet, note that the analytic convergence discussion of the series in section 3, especially formula (7), implies asymptotically geometric convergence of this series, which is the faster, the smaller $|\beta|$ is relatively to α. We suggest to compute the individual summands recursively as discussed below, add them up and stop, when summands become negligible compared to the current value of the approximation. To calculate the individual summands recursively note that

$$a_0(r) = \Gamma\left(\frac{r+1}{2}\right) \tag{22}$$

and

$$a_k(r)t^k = \frac{2\bar{\beta}^2(k + (r-1)/2)}{\bar{\alpha}(2k-1)(2k)} t \cdot a_{k-1}(r) t^{k-1}, \tag{23}$$

which is obtained using the functional equation $\Gamma(z+1) = z\Gamma(z)$ of the Gamma function, and that the recursion formula for Bessel functions (Jørgensen, 1982, p. 170) gives

$$K_{k+\frac{r-1}{2}}(t\bar{\alpha}) = 2 \cdot \left(k - 1 + \frac{r-1}{2}\right)(t\bar{\alpha})^{-1} K_{k-1+\frac{r-1}{2}}(t\bar{\alpha}) + K_{k-2+\frac{r-1}{2}}(t\bar{\alpha}). \tag{24}$$

The latter formula implies that we can calculate the values of the Bessel functions needed from a two-term recursion, for which we only need to calculate $K_{-1+(r-1)/2}(t\bar{\alpha})$ and $K_{(r-1)/2}(t\bar{\alpha})$ as starting values. Hence, the calculation of the value of the series involves, apart from basic manipulations, only one evaluation of the Gamma function and two of the Bessel functions.

The series in the denominator in theorem 3 is the series just discussed above and the numerator is of the same type, only the index of the Bessel functions is changed, and can hence be calculated analogously. Actually, both series can be calculated simultaneously using only the recursion for $a_k(r)t^k$ and the two-term recursion for the Bessel functions described above.

There is, however, one possible problem when using the two-term recursion. If the starting values are zeros up to numerical precision, then only zeros will be calculated as summands. For example when using Matlab and the built in function for K_ν one gets $K_0(z) = 0$ for $z > 697$. Hence, one needs to take care of this possible case. Provided the recursion works, the numerical results obtained are usually almost identical to the numerical results one gets when using a built in Bessel function routine of e.g. Matlab for each summand, but the recursion may save computing power. Furthermore, it should now be obvious, how numerical evaluations of the formulae for μ-centred (absolute) moments of GH laws given in theorem 2 can be organized efficiently.

The Matlab code we used to produce the numerical results in this paper is available from http://www.ma.tum.de/stat/Papers. It is based upon the above considerations and can be used to compute μ-centred moments of the NIG distribution/Lévy process and the derivatives of the log moments with respect to log time.

7. Apparent scaling behaviour of NIG Lévy processes

The aim of this section is to show that NIG Lévy processes may well exhibit a behaviour very close to strict scaling over a wide range of orders of moments. We exemplify the possible apparent scaling of absolute μ-centred moments of NIG Lévy processes using the parameters from Barndorff-Nielsen & Prause (2001). They considered the USD/DEM exchange rate from the whole of 1996, contained in the HFDF96 data set from Olsen & Associates, and fitted an NIG Lévy process to the log returns by maximum likelihood estimation. The estimates obtained based on the 3-hr log returns are $\alpha = 415.9049$, $\beta = 1.512$, $\delta = 0.0011$ and $\mu = 0.000026$. For further details on the data, the estimation procedure and the relevance for finance we refer the interested reader to the paper by Barndorff-Nielsen & Prause. Note especially that, as is typical for returns of exchange rate series, μ is very close to zero and therefore there is practically no difference between moments and μ-centred moments. Figure 2 (left), which depicts the logarithm of the first absolute μ-centred moment versus the logarithm of time in seconds, is therefore optically indistinguishable from the figure in Barndorff-Nielsen & Prause (2001) showing the first absolute moment calculated via numerical integration. The estimated regression line of the log moments against log time, fitted by least squares, has slope 0.5863, which is slightly higher than the slope 0.5705 reported in Barndorff-Nielsen & Prause (2001), and $d \ln E(|Z(t) - \mu t|)/d \ln t$ decreases from 0.7853 to 0.5011 over the time interval depicted, which is 5.625 min to 32 days. This is significantly different from the Brownian motion case, where it is exactly 1/2 (cf. above). The behaviour of $d \ln E(|Z(t) - \mu t|)/d \ln t$ over the time interval considered indicates that for $t \to \infty$ the slope asymptotically becomes about 1/2, the exact Gaussian scaling coefficient. This is related to the fact that $|\beta|$ is relatively small, as already pointed out earlier in the discussion of the scaling asymptotics.

With our results obtained above it is possible to study the behaviour of moments other than the first. Figures 2 (right), 3 (left, right) and 4 (left) show the time behaviour of the 0.9th, 1.1th, 0.5th and 1.5th μ-centred absolute moments over the same time horizon. All figures exhibit apparent scaling, which improves with the order of the moment. The fitted regression lines have slope 0.53535, 0.63536, 0.31322 and 0.81327, respectively, which are all higher than the

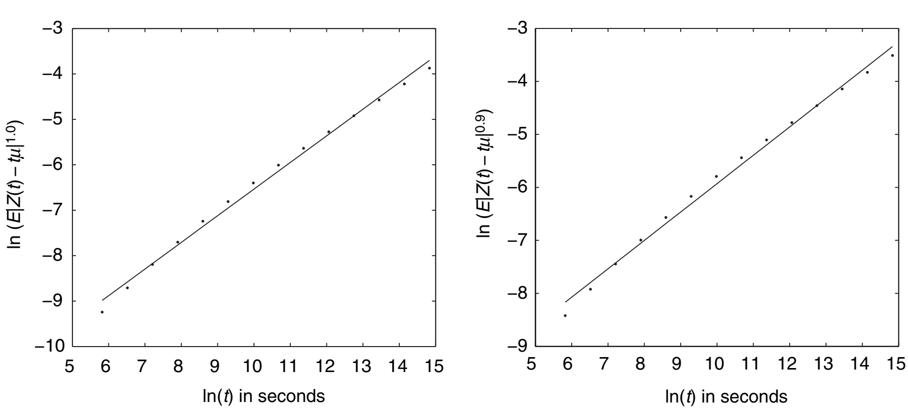

Fig. 2. Approximate scaling law for the first (left) and 0.9th μ-centred absolute moment of the NIG Lévy process fitted to the USD/DEM exchange rate: moments (•) and regression line log moments against log time.

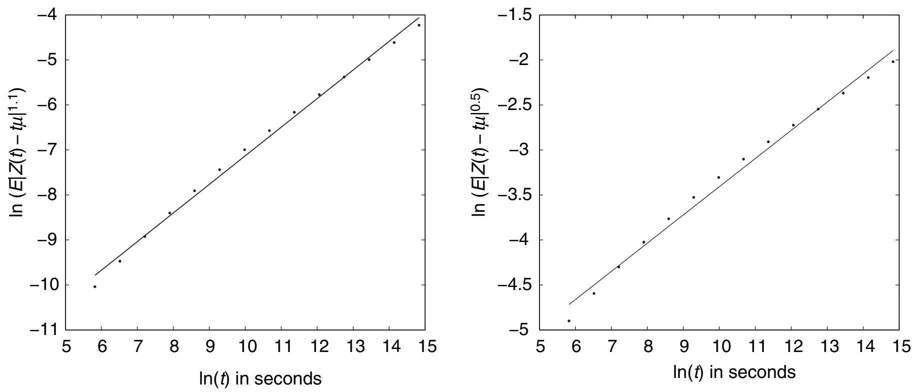

Fig. 3. Approximate scaling law for the 1.1th (left) and 0.5th μ-centred absolute moment: moments (•) and regression line log moments against log time.

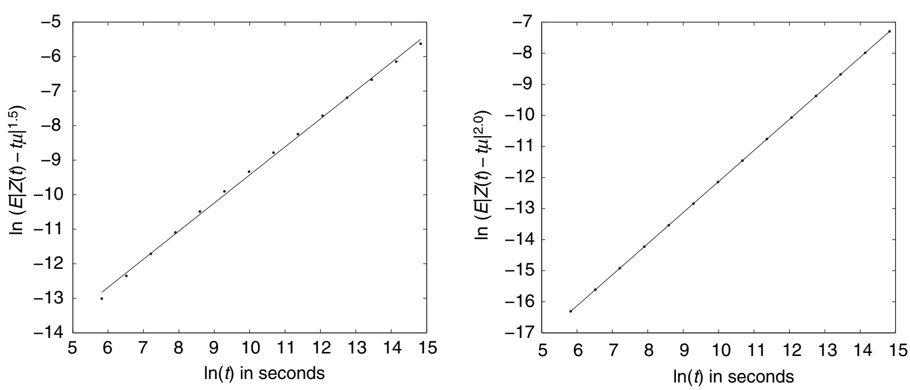

Fig. 4. Approximate scaling law for the 1.5th (left) and second μ-centred absolute moment: moments (•) and regression line log moments against log time.

corresponding values for Brownian motion, which are 0.45, 0.55, 0.25 and 0.75. The values of $d \ln E(|Z(t) - \mu t|^r)/d \ln t$ decrease from 0.7316 to 0.4509, 0.8316 to 0.5512, 0.4499 to 0.2503 and 0.9499 to 0.7515, respectively. So again they seem to converge to some value close to the Brownian motion scaling slope.

Figure 4 (right) shows that the second μ-centred moment seems to exhibit perfect linear scaling. Yet, there is in fact no strict scaling law holding. The values of the regression coefficient 1.0001 and $d \ln E(|Z(t) - \mu t|^2)/d \ln t$ are very close to 1 with $d \ln E(|Z(t) - \mu t|^2)/d \ln t$ increasing very slowly from 1 to 1.0015. Such a result is to be expected, as $|\beta|$ is small (compared with α) and for $\beta = 0$ we have that the variance, which is in this case identical to the second μ-centred moment, obeys a strict scaling law with slope 1, as for Brownian motion.

The third μ-centred absolute moment still exhibits apparent scaling behaviour with a regression slope of 1.2966, but the values of $d \ln E(|Z(t) - \mu t|^3)/d \ln t$ are now increasing from 1.0134 to 1.5007 rather than decreasing and the slope is lower than the scaling coefficient 1.5 for Brownian motion. However, $d \ln E(|Z(t) - \mu t|^3)/d \ln t$ still seems to converge to some value close to 3/2 at large times. It generally seems to be the case that $d \ln E(|Z(t) - \mu t|^r)/d \ln t$ increases with time for $r > 2$, whereas it decreases for $r < 2$. Actually, further calculations indicate that this change takes place marginally below 2 at about 1.9995. Some more numerical calculations hint that in the symmetric case $\ln E(|Z(t) - \mu t|^r)$ is concave as a function of $\ln t$ for $0 < r \leq 2$ and convex for $r \geq 2$.

For very high values of r, i.e. 10 and greater, no apparent linear scaling is observed over the time horizon considered. Looking at the slopes of the apparent scaling of the μ-centred absolute moments (of orders 0.2–4, for instance, as we have done in further calculations), the relationship between scaling coefficient and order is apparently not simply linear as, for example, in the case of an α-stable Lévy process (see Samorodnitsky & Taqqu, 1994, Chapter 7; Sato, 1999, Chapter 3), but a concave one.

Our results obtained above show that NIG Lévy processes may exhibit something close to scaling. These findings are particularly interesting, as both in finance, especially when dealing with foreign exchange returns, and turbulence there is on the one hand empirical and for turbulence also theoretical evidence of scaling laws, and on the other hand models based on NIG Lévy processes have been put forth in the literature. Compared to Brownian motion the NIG Lévy process does in general not exhibit exact linear scaling, but approximate scaling over wide time horizons and for a practically interesting range of (absolute) moments is demonstrated here.

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Appendix A. Moments of GIG and normal laws

For completeness we provide below the well-known formulae for the moments of the GIG and normal laws.

For the GIG law the following result is given in Jørgensen (1982, p. 13), who uses a slightly different parameterization.

Lemma 4
Let $X \sim GIG(\nu, \delta, \gamma)$ with $\delta, \gamma > 0$. Then

$$E(X^r) = \left(\frac{\delta}{\gamma}\right)^r \frac{K_{\nu+r}(\bar{\gamma})}{K_\nu(\bar{\gamma})}$$

for every $r > 0$.
Proof.

$$\begin{aligned} E(X^r) &= \int_0^\infty \frac{\bar{\gamma}^\nu}{2K_\nu(\bar{\gamma})} \delta^{-2\nu} x^{\nu+r-1} \exp\left(-\frac{1}{2}\bar{\gamma}\left((\bar{\gamma}\delta^{-2}x)^{-1} + \bar{\gamma}\delta^{-2}x\right)\right) \mathrm{d}x \overset{y:=\bar{\gamma}\delta^{-2}x}{=} \\ &= \frac{\bar{\gamma}^{-r}\delta^{2r}}{K_\nu(\bar{\gamma})}\frac{1}{2}\int_0^\infty y^{\nu+r-1}\exp\left(-\frac{1}{2}\bar{\gamma}(y^{-1}+y)\right)\mathrm{d}y = \left(\frac{\delta}{\gamma}\right)^r \frac{K_{\nu+r}(\bar{\gamma})}{K_\nu(\bar{\gamma})}, \end{aligned}$$

where in the last step we employed the integral representation (2) of $K_{\nu+r}$ stated earlier when introducing the modified Bessel function of the third kind.

The absolute moments of the normal distribution $N(0, 1)$ are well known and given in many standard texts on probability theory, viz. the following lemma.

Lemma 5
Let $X \sim N(0, 1)$ and $r > 0$ then

$$E(|X|^r) = \frac{2^{r/2}\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi}}.$$

Proof.

$$\begin{aligned} E(|X|^r) &= (2\pi)^{-1/2}\int_{\mathbb{R}} |x|^r \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty x^r \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x \overset{t:=\frac{x^2}{2}}{=} \\ &= \frac{2^{r/2}}{\sqrt{\pi}}\int_0^\infty t^{\frac{r+1}{2}-1}\mathrm{e}^{-t}\,\mathrm{d}t = \frac{2^{r/2}}{\sqrt{\pi}}\Gamma\left(\frac{r+1}{2}\right). \end{aligned}$$

Appendix B. Analyticity of the moments of an NIG Lévy process as a function of time

In this appendix we show that the μ-centred (absolute) moments of an NIG Lévy process Z_t, given in corollary 4, are analytic functions of time. To this end we employ some complex function theory, so we start with a brief review of the needed result, viz. Weierstraß's convergence theorem.

B1. Convergence of sequences of holomorphic functions

Recall that a function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$, is called holomorphic, if it is complex differentiable on D, i.e.

$$\lim_{h\in\mathbb{C}, h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in D$ (confer textbooks on complex function theory, e.g. Remmert, 1991 or Freitag & Busam, 2000, for a thorough discussion of holomorphicity and related concepts).

Holomorphic functions have many useful properties that real differentiable functions lack in general and thus it is often preferable to use holomorphic functions when possible. One of the nice implications of holomorphicity is that any once complex differentiable function is automatically infinitely often complex differentiable and another is that locally uniform convergence commutes with differentiation.

Theorem 4 (*Weierstraß's convergence theorem*)
Let $f_n : D \to \mathbb{C}$, $n \in \mathbb{N}$, be a sequence of holomorphic functions, defined on an open subset $D \subseteq \mathbb{C}$, which converges locally uniform to a function $f : D \to \mathbb{C}$. Then f is holomorphic on D and for every $k \in \mathbb{N}$ the sequence of kth derivatives $f_n^{(k)}$ converges locally uniform to $f^{(k)}$ on D.

For a proof and related results see one of the books mentioned above. The crucial difference to the real differentiable case is that complex differentiation has an integral representation.

B2. Holomorphicity of some series

The following lemma gives in particular that the μ-centred (absolute) moments of an NIG Lévy process Z_t (cf. corollary 4) are analytic functions of time.

Lemma 6
Let $\bar{\alpha} > 0$, $|\bar{\beta}| < \bar{\alpha}$, $1 < \epsilon < \bar{\alpha}^2/|\bar{\beta}|^2$, $v \in \mathbb{R}$, $r > 0$, $n \in \mathbb{N}$, $m = n \bmod 2$, $D = \{z \in \mathbb{C} : \Re(z) > 0,$ $|z| < \epsilon \Re(z)\}$,

$$f : D \to \mathbb{C}, z \mapsto \sum_{k=0}^{\infty} \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{\bar{\alpha}^k (2k)!} z^{k+(r+1)/2} K_{v+k+\frac{r}{2}}(z\bar{\alpha})$$

and

$$g : D \to \mathbb{C}, z \mapsto \sum_{k=0}^{\infty} \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \left\lceil \frac{n}{2} \right\rceil + \frac{1}{2}\right)}{\bar{\alpha}^k (2k+m)!} z^{k+\left\lceil \frac{n}{2} \right\rceil + \frac{1}{2}} K_{v+k+\left\lceil \frac{n}{2} \right\rceil}(z\bar{\alpha}).$$

Then both series are locally uniformly convergent and f, g are holomorphic on D.

Note that v is $-1/2$ in the series of corollary 4.

Proof. It is sufficient to show the locally uniform convergence, since this implies the holomorphicity via Weierstraß's convergence theorem (see appendix B1). Furthermore it is obvious that the result for g follows from the one for f.

Let us now prove the uniform convergence of the series $f(z) = \sum_{k=0}^{\infty} f_k(z)$ on $D \cap \{z \in \mathbb{C} : a < \Re(z) < b\}$ for arbitrary $0 < a < b < \infty$, where

$$f_k(z) = \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{\bar{\alpha}^k (2k)!} z^{k+(r+1)/2} K_{v+k+\frac{r}{2}}(z\bar{\alpha}).$$

An immediate consequence of the integral representation for K_v given in (2) is $|K_v(z)| \leq K_v(\Re(z))$ for $z \in D$ and thus

$$
\begin{aligned}
|f_k(z)| &= \left| \frac{2^k \bar{\beta}^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{\bar{\alpha}^k (2k)!} z^{k+(r+1)/2} K_{v+k+\frac{r}{2}}(z\bar{\alpha}) \right| \\
&\le \frac{2^k |\bar{\beta}|^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{\bar{\alpha}^k (2k)!} (\epsilon \Re(z))^{k+(r+1)/2} K_{v+k+\frac{r}{2}}(\Re(z)\bar{\alpha}) \\
&\overset{x:=\Re(z)}{\le} \frac{2^k |\bar{\beta}|^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{\bar{\alpha}^k (2k)!} (\epsilon x)^{k+(r+1)/2} K_{v+\frac{r}{2}+k}(x\bar{\alpha})
\end{aligned}
$$

for all $k \in \mathbb{N}_0$. Note that we defined $x \in (a, b)$ to be the real part of z. Using equation (17) we obtain

$$
\frac{d}{dx} x^{k+v+\frac{r}{2}} K_{v+\frac{r}{2}+k}(x\bar{\alpha}) = -\bar{\alpha} x^{k+v+\frac{r}{2}} K_{v+\frac{r}{2}+k-1}(x\bar{\alpha}) < 0.
$$

This implies for $x \in (a, b)$:

$$
x^{k+(r+1)/2} K_{v+\frac{r}{2}+k}(x\bar{\alpha}) \le d a^{k+(r+1)/2} K_{v+\frac{r}{2}+k}(a\bar{\alpha})
$$

where $d := a^{v-1/2} \max\{a^{-v+1/2}, b^{-v+1/2}\}$. Applying this inequality to the above expression, we get for all $k \in \mathbb{N}_0$

$$
|f_k(z)| \le \frac{2^k d |\bar{\beta}|^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{\bar{\alpha}^k (2k)!} (\epsilon a)^{k+(r+1)/2} K_{v+\frac{r}{2}+k}(a\bar{\alpha}).
$$

From the finiteness of the $r/2$th absolute moment of the $\mathrm{GH}(v, a\bar{\alpha}, a|\bar{\beta}|\sqrt{\epsilon}, 0, 1)$ law and theorem 2 follows that

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{2^k d |\bar{\beta}|^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{\bar{\alpha}^k (2k)!} (\epsilon a)^{k+(r+1)/2} K_{v+\frac{r}{2}+k}(a\bar{\alpha}) \\
&\quad = d(\epsilon a)^{\frac{r+1}{2}} \sum_{k=0}^{\infty} \frac{2^k (|\bar{\beta}| a \sqrt{\epsilon})^{2k} \Gamma\left(k + \frac{r}{2} + \frac{1}{2}\right)}{(a\bar{\alpha})^k (2k)!} K_{v+\frac{r}{2}+k}(a\bar{\alpha})
\end{aligned}
$$

converges absolutely. Hence, the uniform convergence of $\sum_{k=0}^{\infty} |f_k(z)|$ on $D \cap \{z \in \mathbb{C} : a < \Re(z) < b\}$ is established.