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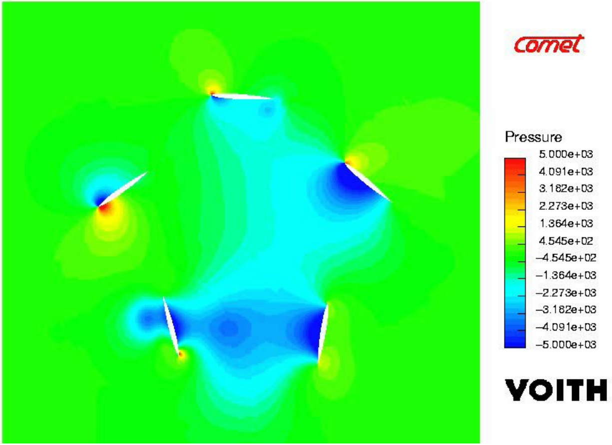

Kristina Steih | Karsten Urban | 15.02.2012

Space-Time Approaches for Reduced Basis Methods

7th Vienna International Conference on Mathematical Modelling

Time-Periodic Problems

Parameter dependent time-periodic problem

Let $V \hookrightarrow H \hookrightarrow V'$ a Gelfand triple, $\mu \in \mathcal{D} \subset \mathbb{R}^p$, $\mathcal{A}(t;\mu) : V \to V'$ uniformly continuous and coercive.

$$J(u(\mu)) = \int_0^T \ell(u(\mu);\mu)\,dt,$$

$$u_t + \mathcal{A}(t;\mu)u = g(t;\mu) \qquad \text{on } \Omega,$$

$$u(0) = u(T) \quad \text{in } H,$$

$$u(t,x) = h_D(x) \text{ on } \Gamma_D(\mu), \quad \frac{\partial}{\partial n}u(t,x) = h_N(x) \text{ on } \Gamma_N(\mu).$$

Reduced Basis Method:

- Basis of snapshots (“truth” solutions)
- A-posteriori error bounds
- Online-offline decomposition

Some necessary notation

Basis dimensions:

$u(\mu) \in \mathcal{X}$ Function Space

Discretization

$u^{\mathcal{N}}(\mu) \in \mathcal{X}^{\mathcal{N}}$ Truth Approximation

Reduction

$u_N(\mu) \in \mathcal{X}_N$ RB Approximation

Function spaces:

$$
\begin{aligned}
\mathcal{X}^{per} &:= L_2(0, T; V) \cap H^1_{per}(0, T; V') \\
&:= \{u \in L_2(0, T; V) : u_t \in L_2(0, T; V'), u(0) = u(T) \text{ in } H\}, \\
\mathcal{Y} &:= L_2(0, T; V)
\end{aligned}
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$$e_N(\mu) = u^{\mathcal{N}}(\mu) - u_N(\mu)$$

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Fixed Point Method

Space-Time Method

Wavelet Basis

Numerical Example

Fixed Point Method

Space-Time Method

Wavelet Basis

Numerical Example

Fixed-point approach

Variational Formulation

Find $u(\mu) \in \mathcal{X}$ with $u(0;\mu) = u(T;\mu)$ and

$$\langle u_t, v\rangle + a(t,u,v;\mu) = g(t,v;\mu) \qquad \forall\, v \in V, t \in [0,T].$$

Fixed-point approach

Variational Formulation (Implicit Euler)

For $\{t_k\}_{k=1,\dots,K}$, find $u^k(\mu) = u(t_k;\mu) \in V$ with $u^0(\mu) = u^K(\mu)$ and

$$\langle u^k, v\rangle + \Delta t\, a(t_k, u^k, v; \mu) = \langle u^{k-1}, v\rangle + \Delta t\, g(t_k, v; \mu) \qquad \forall\, v \in V.$$

- **Periodicity:** Fixed-point iterations $\|u^K_{(M)}(\mu) - u^0_{(M)}(\mu)\|_V \leq tol$

Fixed-point approach

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Time-Discrete Error Bound

$$\|e_N(\mu)\|_{\mathcal{Y}} \approx \left(\Delta t \sum_{k=1}^{K} \|e_N^k(\mu)\|_V^2\right)^{\frac{1}{2}} \leq \left(\frac{\Delta t}{\alpha^2(\mu)} \sum_{k=1}^{K} \|r_N^k(\cdot;\mu)\|_{V'}^2\right)^{\frac{1}{2}} =: \Delta_N^{\mathsf{FP},\mathcal{Y}}(\mu).$$

- $r_N^k(\mu) : V \to \mathbb{R}$ residual at time step t_k
- $\alpha(\mu)$ coercivity constant of $a(t, \cdot, \cdot; \mu)$.

Advantages / Disadvantages

- Fixed-Point iterations
  - + Easily computable
  - + Low memory requirement
  - – Number M of necessary iterations unknown a priori, may be large
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- Time-dependent operators (LTV problems):
  - – Additional offline quantities (at each time step) or
  - – EIM in time (Grepl 2011)
  - – Computation of $\alpha(\mu) = \min_{t\in[0,T]} \alpha(t;\mu)$?

Space-Time approach

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Space-Time approach

Variational Formulation

Find $u(\mu) \in \mathcal{X}^{per}$ with

$$\int_0^T \langle u_t, v \rangle \, dt + \int_0^T a(t, u, v; \mu) \, dt = \int_0^T g(t, v; \mu) \, dt \qquad \forall \, v \in \mathcal{Y}.$$

Space-Time approach

Variational Formulation

Find $u(\mu) \in \mathcal{X}^{per}$ with

$$\underbrace{\int_0^T \langle u_t, v\rangle \, dt + \int_0^T a(t,u,v;\mu)\, dt}_{=: b(u,v;\mu)} = \underbrace{\int_0^T g(t,v;\mu)\, dt}_{:= f(v;\mu)} \qquad \forall\, v \in \mathcal{Y}.$$

Space-Time approach

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Space-Time approach

Variational Formulation

Find $u(\mu) \in \mathcal{X}^{per}$ with

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Error Bounds

$$\|e_N(\mu)\|_{\mathcal{Y}} \;\leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\alpha(\mu)} =: \Delta_N^{\mathsf{ST},\mathcal{Y}}(\mu),$$

- $r_N(\mu) : \mathcal{Y} \to \mathbb{R}$ space-time residual

Space-Time approach

Variational Formulation

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$$\|e_N(\mu)\|_{\mathcal{Y}} \leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\alpha(\mu)} =: \Delta_N^{\mathsf{ST},\mathcal{Y}}(\mu),$$

$$\|e_N(\mu)\|_{\mathcal{X}} \leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\beta(\mu)} =: \Delta_N^{\mathsf{ST},\mathcal{X}}(\mu).$$

- $r_N(\mu) : \mathcal{Y} \to \mathbb{R}$ space-time residual
- $\beta(\mu) = \inf_{0 \neq u \in \mathcal{X}^{per}} \sup_{0 \neq v \in \mathcal{Y}} \frac{|b(u,v;\mu)|}{\|u\|_{\mathcal{X}^{per}} \|v\|_{\mathcal{Y}}}$ inf-sup constant of $b(\cdot, \cdot; \mu)$.

Advantages / Disadvantages

- Space-Time formulation:
  - + Periodicity in basis, no fixed-point iterations necessary
  - + Error bounds time-continuous, in $\mathcal{X}$ as well as in $\mathcal{Y}$
  - – High memory requirement due to additional dimension
  - – Riesz representors, inf-sup constants: space-time problems

Fixed-Point:

- Time-discrete error bound in $\mathcal{Y}$
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  - – High memory requirement due to additional dimension
  - – Riesz representors, inf-sup constants: space-time problems
- Greedy to construct *space-time basis* $\mathcal{X}_N \subset \mathcal{X}$
  - \+ Online: 1 linear system with computational effort $\mathcal{O}(N_{\text{ST}}^3)$
  - – High memory requirement for space-time basis functions
- Time-dependent operators (LTV problems)
  - \+ easily treatable

Fixed-Point:

- Time-discrete error bound in $\mathcal{Y}$
- Online: $\mathcal{O}(MKN_{\text{FP}}^3)$

Fixed Point Method

Space-Time Method

Wavelet Basis

Numerical Example

Truth Discretization

Tensor Basis Structure:

$$\mathcal{X}^{per} = (L_2(0,T) \otimes V) \cap (H^1_{per}(0,T) \otimes V'),$$
$$\mathcal{Y} = L_2(0,T) \otimes V.$$

- Approximation by tensor product of discrete spaces

Why Wavelets?

- Easy construction of periodic wavelets
- Existence of space-time adaptive algorithms [Schwab, Stevenson (2009)] with optimal convergence rate to avoid penalty of additional dimension

  Formulation as equivalent bi-infinite matrix-vector problem:

$$b(u,v;\mu) = f(v;\mu), \quad u \in \mathcal{X}, v \in \mathcal{Y} \quad \Longleftrightarrow \quad \mathbf{Bu} = \mathbf{f}, \qquad u, f \in \ell_2$$

Fixed Point Method

Space-Time Method

Wavelet Basis

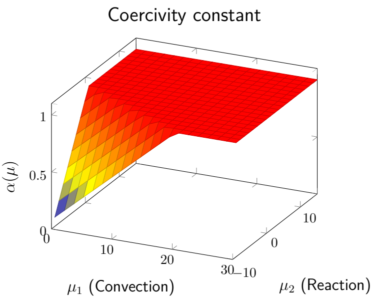
Numerical Example

Convection-Diffusion-Reaction Example

CDR Problem:

$$u_t - u_{xx} + \mu_1(\tfrac{1}{2} - x)u_x + \mu_2 u = \cos(2\pi t) \qquad \text{on } \Omega = (0,1),$$
$$u(t,0) = u(t,1) = 0,$$
$$u(0,x) = u(T,x), \qquad (T = 1).$$

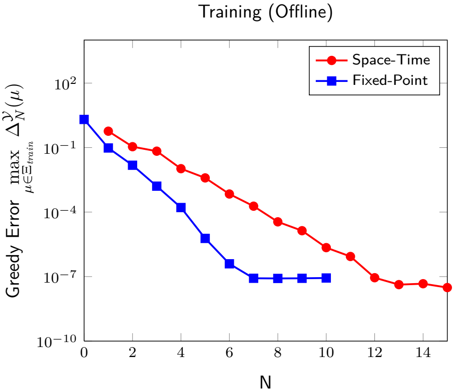
Parameter Domain: $\mu \in \mathcal{D} := [0,30] \times [-9,15]$.



Implementations:

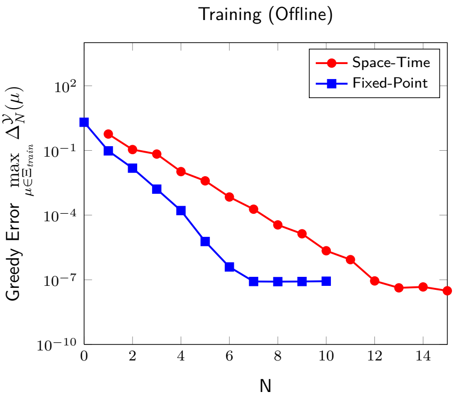
- Space-Time: `LAWA` (Library for Adaptive Wavelet Applications)
- Fixed-Point: `rbOOmit/libmesh`

Training and Error Bound Quality



(a) Training ($n_{\text{train}} = 400$)

Training and Error Bound Quality



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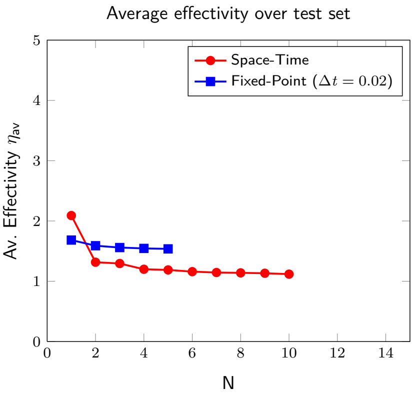
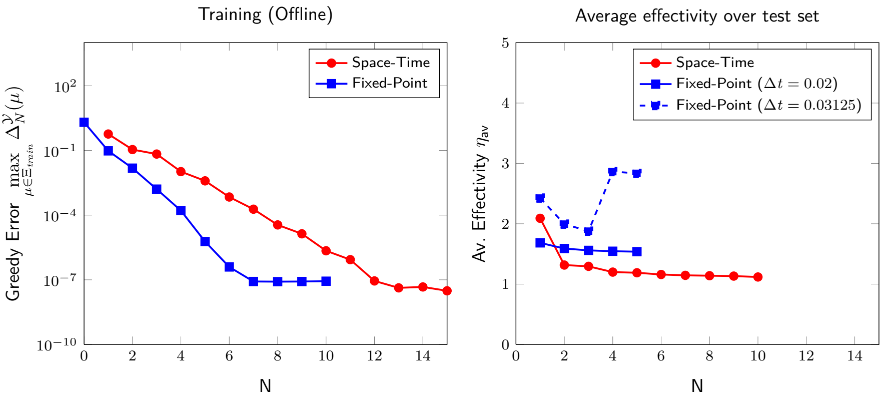


(b) Average Effectivity ($n_{\text{test}} = 225$)

$$\eta_{\text{av}} := \frac{1}{n_{\text{test}}} \sum_{\mu \in \Xi_{\text{test}}} \frac{\Delta_N^{\mathcal{Y}}(\mu)}{\|e_N(\mu)\|}$$

Training and Error Bound Quality



(a) Training ($n_{\text{train}} = 400$)   (b) Average Effectivity ($n_{\text{test}} = 225$)

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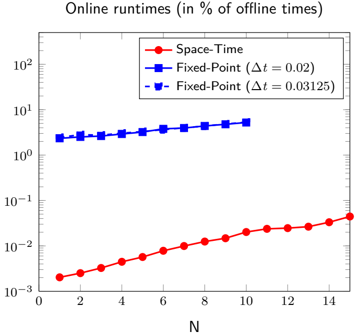
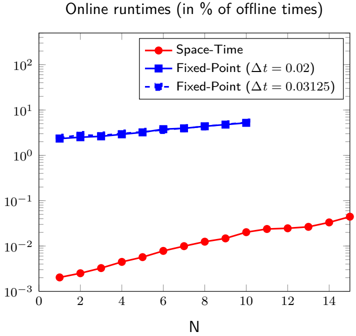
Runtime Reduction



(a) Runtime reduction

	Time Truth (in s)
Space-Time	0.8965
Fixed-Point ($\Delta t = 0.02$)	0.1076
Fixed-Point ($\Delta t = 0.03125$)	0.0662

Runtime Reduction



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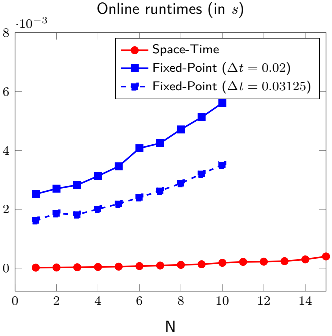

(b) Online runtimes

	Time Truth (in s)
Space-Time	0.8965
Fixed-Point ($\Delta t = 0.02$)	0.1076
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Conclusion

Fixed-Point Approach:

- Low memory requirement
- Convergence uncertain
- Discrete error bound in $\mathcal{Y}$
- Online: $\mathcal{O}(MKN_{\mathsf{FP}}^3)$

Space-Time Approach:

- Additional dimension
- No fixed-point iterations
- Continuous error bounds in $\mathcal{X}$ and $\mathcal{Y}$
- Online: $\mathcal{O}(N_{\mathsf{ST}}^3)$

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Questions?

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Wavelet Tensor Basis

1D Bases:

(i) Space Discretization: Collection $\Sigma \subset V$

- normalized Riesz basis of H, homogeneous boundary conditions
- renormalization possible to obtain Riesz bases of V and V'

(ii) Time Discretization: Collection $\Theta \subset L_2(0,T)$

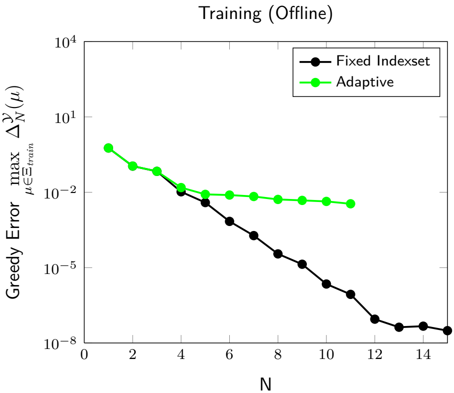
- periodic basis functions, normalized in $L_2(0,T)$
- renormalization possible to obtain Riesz basis of $H^1(0,T)$

2D Basis:

Collection $\Theta \otimes \Sigma = \{(t,x) \mapsto \theta_{\lambda_\Theta}(t)\sigma_{\lambda_\Sigma}(x) \ : \ \lambda_\Theta \in \mathcal{I}_\Theta, \lambda_\Sigma \in \mathcal{I}_\Sigma\}$

- Basis of $\mathcal{X}^{per}$: $\left\{(t,x) \mapsto \frac{\theta_{\lambda_\Theta}(t)\sigma_{\lambda_\Sigma}(x)}{\sqrt{\|\theta_{\lambda_\Theta}\|^2_{\mathbf{L_2(0,T)}}\|\sigma_{\lambda_\Sigma}\|^2_{\mathbf{V}} + \|\theta_{\lambda_\Theta}\|^2_{\mathbf{H^1(0,T)}}\|\sigma_{\lambda_\Sigma}\|^2_{\mathbf{V'}}}}\right\}$
- Basis of $\mathcal{Y}$: $\left\{(t,x) \mapsto \frac{\theta_{\lambda_\Theta}(t)\sigma_{\lambda_\Sigma}(x)}{\sqrt{\|\theta_{\lambda_\Theta}\|^2_{\mathbf{L_2(0,T)}}\|\sigma_{\lambda_\Sigma}\|^2_{\mathbf{V}}}}\right\}$

Adaptive Approach



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