Space-Time Approaches for Reduced Basis Methods

Time-Periodic Problems
Parameter dependent time-periodic problem

Let $V \hookrightarrow H \hookrightarrow V'$ a Gelfand triple, $\mu \in \mathcal{D} \subset \mathbb{R}^p$, $A(t; \mu) : V \rightarrow V'$ uniformly continuous and coercive.

$$J(u(\mu)) = \int_0^T \ell(u(\mu); \mu) \, dt,$$

$$u_t + A(t; \mu) u = g(t; \mu) \quad \text{on } \Omega,$$

$$u(0) = u(T) \quad \text{in } H,$$

$$u(t, x) = h_D(x) \quad \text{on } \Gamma_D(\mu), \quad \frac{\partial}{\partial n} u(t, x) = h_N(x) \quad \text{on } \Gamma_N(\mu).$$

**Reduced Basis Method:**

- Basis of snapshots ("truth" solutions)
- A-posteriori error bounds
- Online-offline decomposition
Some necessary notation

Basis dimensions:

\[ u(\mu) \in \mathcal{X} \quad \text{Function Space} \]

Discretization

\[ u^N(\mu) \in \mathcal{X}^N \quad \text{Truth Approximation} \]

Reduction

\[ u_N(\mu) \in \mathcal{X}_N \quad \text{RB Approximation} \]

Function spaces:

\[ \mathcal{X}^{\text{per}} := L^2(0, T; V) \cap H^1_{\text{per}}(0, T; V') \]

\[ := \{ u \in L^2(0, T; V) : u_t \in L^2(0, T; V'), u(0) = u(T) \text{ in } H \}, \]

\[ \mathcal{Y} := L^2(0, T; V) \]
Some necessary notation

Basis dimensions:

\[ u(\mu) \in \mathcal{X} \quad \text{Function Space} \]
\[ \downarrow \quad \text{Discretization} \]
\[ u^{\mathcal{N}}(\mu) \in \mathcal{X}^{\mathcal{N}} \quad \text{Truth Approximation} \]
\[ \downarrow \quad \text{Reduction} \]
\[ u_N(\mu) \in \mathcal{X}_N \quad \text{RB Approximation} \]

\[ e_N(\mu) = u^{\mathcal{N}}(\mu) - u_N(\mu) \]

Function spaces:

\[ \mathcal{X}^{\text{per}} := L_2(0, T; V) \cap H_{\text{per}}^1(0, T; V') \]
\[ := \{ u \in L_2(0, T; V) : u_t \in L_2(0, T; V'), u(0) = u(T) \text{ in } H \} \],
\[ \mathcal{Y} := L_2(0, T; V) \]
Fixed Point Method

Space-Time Method

Wavelet Basis

Numerical Example
Fixed Point Method

Space-Time Method

Wavelet Basis

Numerical Example
Fixed-point approach

Variational Formulation

Find $u(\mu) \in \mathcal{X}$ with $u(0; \mu) = u(T; \mu)$ and

$$\langle u_t, v \rangle + a(t, u, v; \mu) = g(t, v; \mu) \quad \forall v \in V, t \in [0, T].$$
Fixed-point approach

Variational Formulation (Implicit Euler)

For $\{t_k\}_{k=1,...,K}$, find $u^k(\mu) = u(t_k; \mu) \in V$ with $u^0(\mu) = u^K(\mu)$ and

$$\langle u^k, v \rangle + \Delta t a(t_k, u^k, v; \mu) = \langle u^{k-1}, v \rangle + \Delta t g(t_k, v; \mu) \quad \forall \, v \in V.$$ 

- **Periodicity**: Fixed-point iterations $\|u^K_M(\mu) - u^0_M(\mu)\|_V \leq tol$
Fixed-point approach

Variational Formulation

For \( \{t_k\}_{k=1,...,K} \), find \( u^k(\mu) = u(t_k; \mu) \in V \) with \( u^0(\mu) = u^K(\mu) \) and

\[
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\]

- **Periodicity:** Fixed-point iterations \( \| u^K_M(\mu) - u^0_M(\mu) \|_V \leq tol \)

Time-Discrete Error Bound

\[
\| e_N(\mu) \|_Y \approx \left( \Delta t \sum_{k=1}^{K} \| e^k_N(\mu) \|_V^2 \right)^{\frac{1}{2}} \leq \left( \frac{\Delta t}{\alpha^2(\mu)} \sum_{k=1}^{K} \| r^k_N(\cdot; \mu) \|_V^2 \right)^{\frac{1}{2}} =: \Delta_{FP,Y}^N(\mu).
\]

- \( r^k_N(\mu) : V \rightarrow \mathbb{R} \) residual at time step \( t_k \)
- \( \alpha(\mu) \) coercivity constant of \( a(t, \cdot, \cdot; \mu) \).
Advantages / Disadvantages

- Fixed-Point iterations
  - Easily computable
  - Low memory requirement
  - Number $M$ of necessary iterations unknown a priori, may be large
  - Choice of initial value is critical
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- POD-Greedy to construct *time-independent* basis $V_N \subset V$
  - Low memory requirement
  - Calculations of Riesz representors and coercivity constant are problems in $V$ only
  - Reduction only in space
  - Time discretization fixed a priori, error bound discrete
  - Online: fixed-point iteration with computational effort $\mathcal{O}(MKN_{FP}^3)$
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- **Fixed-Point iterations**
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- **POD-Greedy to construct time-independent basis \( V_N \subset V \)**
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- **Time-dependent operators (LTV problems):**
  - Additional offline quantities (at each time step) or
  - EIM in time (Grepl 2011)
  - Computation of \( \alpha(\mu) = \min_{t \in [0, T]} \alpha(t; \mu) \) ?
Fixed Point Method

Space-Time Method

Wavelet Basis

Numerical Example
Space-Time approach

Variational Formulation

Find $u(\mu) \in \mathcal{X}$ with $u(0; \mu) = u(T; \mu)$ and

$$\langle u_t, v \rangle + a(t, u, v; \mu) = g(t, v; \mu) \quad \forall \, v \in V, \, t \in [0, T].$$
Space-Time approach

Variational Formulation

Find $u(\mu) \in \mathcal{X}^{per}$ with

$$\int_0^T \langle u_t, v \rangle dt + \int_0^T a(t, u, v; \mu) dt = \int_0^T g(t, v; \mu) dt \quad \forall v \in \mathcal{Y}. $$
Space-Time approach

Variational Formulation

Find $u(\mu) \in X^{per}$ with

$$
\int_0^T \langle u_t, v \rangle dt + \int_0^T a(t, u, v; \mu) dt = \int_0^T g(t, v; \mu) dt
$$

$= b(u,v;\mu)$

$\forall v \in Y.$

$:\:= f(v;\mu)$
Space-Time approach

Variational Formulation

Find $u(\mu) \in X^{per}$ with

$$b(u, v; \mu) = f(v; \mu) \quad \forall v \in Y.$$
Space-Time approach

Variational Formulation

Find $u(\mu) \in X^{per}$ with

$$b(u, v; \mu) = f(v; \mu) \quad \forall v \in Y.$$ 

Error Bounds

$$\|e_N(\mu)\|_Y \leq \frac{\|r_N(\cdot; \mu)\|_{Y'}}{\alpha(\mu)} =: \Delta^{\text{ST}, Y}_N(\mu),$$

$\triangleright r_N(\mu) : Y \to \mathbb{R}$ space-time residual
Space-Time approach

Variational Formulation

Find \( u(\mu) \in X^{per} \) with

\[
b(u, v; \mu) = f(v; \mu) \quad \forall v \in Y.
\]

Error Bounds

\[
\| e_N(\mu) \|_Y \leq \frac{\| r_N(\cdot; \mu) \|_{Y'}}{\alpha(\mu)} =: \Delta^{ST,Y}_N(\mu),
\]

\[
\| e_N(\mu) \|_X \leq \frac{\| r_N(\cdot; \mu) \|_{Y'}}{\beta(\mu)} =: \Delta^{ST,X}_N(\mu).
\]

- \( r_N(\mu) : Y \to \mathbb{R} \) space-time residual
- \( \beta(\mu) = \inf_{0 \neq u \in X^{per}} \sup_{0 \neq v \in Y} \frac{|b(u, v; \mu)|}{\| u \|_{X^{per}} \| v \|_Y} \) inf-sup constant of \( b(\cdot, \cdot; \mu). \)
Advantages / Disadvantages

▶ Space-Time formulation:
  + Periodicity in basis, no fixed-point iterations necessary
  + Error bounds time-continuous, in $\mathcal{X}$ as well as in $\mathcal{Y}$
  − High memory requirement due to additional dimension
  − Riesz representors, inf-sup constants: space-time problems

Fixed-Point:
▶ Time-discrete error bound in $\mathcal{Y}$
▶ Online: $O(MKN_{\text{FP}}^3)$
Advantages / Disadvantages

▶ Space-Time formulation:
   + Periodicity in basis, no fixed-point iterations necessary
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▶ Greedy to construct \textit{space-time basis} \( \mathcal{X}_N \subset \mathcal{X} \)
   + Online: 1 linear system with computational effort \( \mathcal{O}(N_{ST}^3) \)
   − High memory requirement for space-time basis functions

▶ Time-dependent operators (LTV problems)
   + easily treatable

Fixed-Point:
▶ Time-discrete error bound in \( \mathcal{Y} \)
▶ Online: \( \mathcal{O}(MKN_{FP}^3) \)
Fixed Point Method

Space-Time Method

Wavelet Basis

Numerical Example
Truth Discretization

Tensor Basis Structure:

\[ X^{per} = (L_2(0, T) \otimes V) \cap (H^1_{per}(0, T) \otimes V'), \]
\[ Y = L_2(0, T) \otimes V. \]

- Approximation by tensor product of discrete spaces

Why Wavelets?

- Easy construction of periodic wavelets
- Existence of space-time adaptive algorithms [Schwab, Stevenson (2009)] with optimal convergence rate to avoid penalty of additional dimension

Formulation as equivalent bi-infinite matrix-vector problem:

\[ b(u, v; \mu) = f(v; \mu), \quad u \in X, v \in Y \iff \quad Bu = f, \quad u, f \in \ell_2 \]
Fixed Point Method

Space-Time Method

Wavelet Basis

Numerical Example
Convection-Diffusion-Reaction Example

CDR Problem:

\[ u_t - u_{xx} + \mu_1 \left( \frac{1}{2} - x \right) u_x + \mu_2 u = \cos(2\pi t) \quad \text{on } \Omega = (0, 1), \]

\[ u(t, 0) = u(t, 1) = 0, \]

\[ u(0, x) = u(T, x), \quad (T = 1). \]

Parameter Domain: \( \mu \in \mathcal{D} := [0, 30] \times [-9, 15]. \)

Implementations:
- Space-Time: LAWA (Library for Adaptive Wavelet Applications)
- Fixed-Point: rb00mit/libmesh
Training and Error Bound Quality

(a) Training \( (n_{\text{train}} = 400) \)
Training and Error Bound Quality

Training (Offline)

Average effectivity over test set

(a) Training \( n_{\text{train}} = 400 \)

(b) Average Effectivity \( n_{\text{test}} = 225 \)

\[
\eta_{\text{av}} := \frac{1}{n_{\text{test}}} \sum_{\mu \in \Xi_{\text{test}}} \frac{\Delta^V_N(\mu)}{\|e_N(\mu)\|}
\]
Training and Error Bound Quality

(a) Training \((n_{\text{train}} = 400)\)

(b) Average Effectivity \((n_{\text{test}} = 225)\)

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Runtime Reduction

Online runtimes (in % of offline times)

(a) Runtime reduction

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Conclusion

**Fixed-Point Approach:**
- Low memory requirement
- Convergence uncertain
- Discrete error bound in $\mathcal{Y}$
- Online: $\mathcal{O}(MKN_{FP}^3)$

**Space-Time Approach:**
- Additional dimension
- No fixed-point iterations
- Continuous error bounds in $\mathcal{X}$ and $\mathcal{Y}$
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Questions?
References

M.A. Grepl.

LAWA (2011).
(Library for Adaptive Wavelet Applications). lawa.sourgeforge.net.


C. Schwab and R. Stevenson.
Wavelet Tensor Basis

1D Bases:

(i) Space Discretization: Collection $\Sigma \subset V$
   - normalized Riesz basis of $H$, homogeneous boundary conditions
   - renormalization possible to obtain Riesz bases of $V$ and $V'$

(ii) Time Discretization: Collection $\Theta \subset L_2(0, T)$
   - periodic basis functions, normalized in $L_2(0, T)$
   - renormalization possible to obtain Riesz basis of $H^1(0, T)$

2D Basis:
Collection $\Theta \otimes \Sigma = \{ (t, x) \mapsto \theta_{\lambda\Theta}(t)\sigma_{\lambda\Sigma}(x) : \lambda\Theta \in \mathcal{I}_\Theta, \lambda\Sigma \in \mathcal{I}_\Sigma \}$

- Basis of $\mathcal{X}_{\text{per}}$:
  \[
  (t, x) \mapsto \frac{\theta_{\lambda\Theta}(t)\sigma_{\lambda\Sigma}(x)}{\sqrt{\|\theta_{\lambda\Theta}\|_{L_2(0,T)}^2\|\sigma_{\lambda\Sigma}\|^2_V + \|\theta_{\lambda\Theta}\|_{H^1(0,T)}^2\|\sigma_{\lambda\Sigma}\|^2_{V'}}}
  \]

- Basis of $\mathcal{Y}$:
  \[
  (t, x) \mapsto \frac{\theta_{\lambda\Theta}(t)\sigma_{\lambda\Sigma}(x)}{\sqrt{\|\theta_{\lambda\Theta}\|_{L_2(0,T)}^2\|\sigma_{\lambda\Sigma}\|^2_V}}
  \]
Adaptive Approach

![Graph showing Greedy Error](image)

(a) Training ($n_{\text{train}} = 400$)