ulm university universität uulm

Fakultät für Mathematik und Wirtschaftswissenschaften

Institut für Numerische Mathematik

Bachelorarbeit

Efficient Implementation of the hp-Boundary Element Method for the Navier-Lamé Equation

vorgelegt von

Andreas Bantle

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Gutachter

Prof. Dr. S. Funken

Vorwort.

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Kapitel 1

Full Title

1.1 Introduction

```
function [value,I] = detDFmin(coordinates,elements)
%*** Compute min_{x\in T} |det(DF)| / (2*|T|) for all triangles
%
value = Inf(size(elements,1),7);
C = [0,1;-1,0];
succ = [2,3,1];
for j=1:size(elements,1)
  coord = coordinates(elements(j,:),:);
  A = coord([1,4,2,5],:)-coord([2,1,4,6],:);
  B = coord([1,6,5,3],:)-coord([3,1,4,6],:);
  %*** Compte min(DF) for each vertex value(:,1:3) and edge value(:,4:6)
  for k=1:3
    U = (A(1,:)  +4*A(1+succ(k),:)) * C * (B(1,:)  +4*B(1+succ(k),:))';
    V = (A(1+k,:)-  A(1+succ(k),:)) * C * (B(1,:)  +4*B(1+succ(k),:))' ...
      + (A(1,:)  +4*A(1+succ(k),:)) * C * (B(1+k,:)-  B(1+succ(k),:))';
    W(k) = (A(1+k,:)-  A(1+succ(k),:)) * C * (B(1+k,:)-  B(1+succ(k),:))';
    value(j,k) = U;
    if W(k) > 0 && abs(V+4*W(k)) < abs(4*W(k))
      value(j,k+3) = (U*W(k)-0.25*V^2)/W(k);
    end
  end
  %*** compute min(DF) in interior if Laplace(DF) is positive
  if W(1) > 0
    W(4) = (A(3,:)-A(2,:)) * C * (B(4,:)-B(2,:))' ...
         + (A(4,:)-A(2,:)) * C * (B(3,:)-B(2,:))';
    det = (4*W(1)*W(3)-W(4)^2);
    if det > 0
      b = [(A(3,:)-A(2,:)) * C * (B(1,:)+B(2,:))' ...
          +(A(1,:)+A(2,:)) * C * (B(3,:)-B(2,:))';...
           (A(4,:)-A(2,:)) * C * (B(1,:)+B(2,:))' ...
          +(A(1,:)+A(2,:)) * C * (B(4,:)-B(2,:))'];
      X = -1/det*[2*W(3),-W(4);-W(4),2*W(1)]*b;
      if X(1) > 0 && X(2) > 0 && X(1)+X(2) < 1
        value(j,7) = (A(1,:)+A(2,:))*C*(B(1,:)+B(2,:))' + 1/2*X'*b;
      end
    end
  end
end
[value,I]=min(value,[],2);
%*** Compute area
[bary,wg] = quad2d(6);
phi  = [bary.*(2*bary-1),4*bary.*bary(:,[2,3,1])];
D = [-1,1,0;-1,0,1];
C = reshape(coordinates(elements',:),6,[]);
area = zeros(size(elements,1),1);
for k = 1:size(wg,1)
  dphi = [D.*(4*bary([k,k],:)-1), ...
          4*(D.*bary([k,k],[2,3,1])+D(:,[2,3,1]).*bary([k,k],:))];
```

```
  DF = reshape((dphi*C)',[],4);
  area =  area + wg(k)/2 * (DF(:,1).*DF(:,4)-DF(:,2).*DF(:,3));
end
value = value./(2*area);
```

Anhang A
Calculation of the Integral $O^1_{j,k}$

Lemma A.0.1 *Let $a, b \in \mathbb{C}$, with $a \pm b \notin [-1,1]$, $j \geq 2$ and η_1 as defined in (??). Then, there holds*

$$\begin{aligned}
O^1_{0,3}(a,b) = {} & a\left[\widetilde{Q}_0^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_0^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right] + b\left[\widetilde{Q}_1^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_1^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right] \\
& - \frac{1}{2}\left(\frac{a^2-1}{b} - \frac{1}{3}b\right)\left[\widetilde{Q}_0^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_0^0\left(\frac{-1-a}{b}\right)\right] \\
& - a\left[\widetilde{Q}_1^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^0\left(\frac{-1-a}{b}\right)\right] - \frac{1}{3}b\left[\widetilde{Q}_2^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_2^0\left(\frac{-1-a}{b}\right)\right] + 2,
\end{aligned}$$

$$\begin{aligned}
O^1_{1,3}(a,b) = {} & a\left[\widetilde{Q}_1^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_1^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right] + \frac{2}{3}b\left[\widetilde{Q}_2^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_2^{-1}\left(\frac{-1-a}{b}\right)\right] \\
& + \frac{1}{3}b\left[\widetilde{Q}_0^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_0^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right] \\
& - \frac{1}{2}\left(\frac{a^2-1}{b} - \frac{3}{5}b\right)\left[\widetilde{Q}_1^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^0\left(\frac{-1-a}{b}\right)\right] \\
& - \frac{2}{3}a\left[\widetilde{Q}_2^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_2^0\left(\frac{-1-a}{b}\right)\right] - \frac{1}{3}a\left[\widetilde{Q}_0^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_0^0\left(\frac{-1-a}{b}\right)\right] \\
& - \frac{1}{5}b\left[\widetilde{Q}_3^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_3^0\left(\frac{-1-a}{b}\right)\right] + 2.
\end{aligned}$$

and

$$\begin{aligned}
O^1_{j,3}(a,b) = {} & a\,\frac{j}{(j+2)(2j+1)}\left\{\widetilde{Q}_{j+1}^0\left(\frac{-1-a}{b}\right) - \widetilde{Q}_{j+1}^0\left(\frac{1-a}{b}\right)\right\} \\
& + \left[\frac{a^2-1}{b}\,\frac{1}{j+2} + b\,\frac{j-1}{(j+2)(2j-1)}\right]\left\{\widetilde{Q}_j^0\left(\frac{-1-a}{b}\right) - \widetilde{Q}_j^0\left(\frac{1-a}{b}\right)\right\} \\
& + a\,\frac{3j+2}{(j+2)(2j+1)}\left\{\widetilde{Q}_{j-1}^0\left(\frac{-1-a}{b}\right) - \widetilde{Q}_{j-1}^0\left(\frac{1-a}{b}\right)\right\} \\
& + b\,\frac{j}{(j+2)(2j-1)}\left\{\widetilde{Q}_{j-2}^0\left(\frac{-1-a}{b}\right) - \widetilde{Q}_{j-2}^0\left(\frac{1-a}{b}\right)\right\},
\end{aligned}$$

where η_2 is defined as in Lemma ??.

Beweis. We start proving the first identity. Using the definition of $N_3(t)$, we have

$$O^1_{0,3}(a,b) = \frac{1}{2}\left(\int_{-1}^{1}\int_{-1}^{1}\frac{t^2}{(a+b\,s-t)^2}\,ds\,dt - \int_{-1}^{1}\int_{-1}^{1}\frac{1}{(a+b\,s-t)^2}\,ds\,dt\right). \quad \text{(A.1)}$$

Therefore, we investigate both integrals separately. There holds

$$
\begin{aligned}
\int_{-1}^{1} \frac{t^2}{(a+b\,s-t)^2}\,dt &= t + 2\,(a+b\,s)\,\log(-a-b\,s+t)\Big|_{-1}^{1} \\
&\quad + \frac{(a+b\,s)^2}{a+b\,s-t}\Big|_{-1}^{1} \\
&= 2 + 2(a+b\,s)\,\left[\log(-a-b\,s+1) - \log(-a-b\,s-1)\right] \\
&\quad + (a+b\,s)^2\left[\frac{1}{a+b\,s-1} - \frac{1}{a+b\;+1}\right].
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1} \frac{t^2}{(a+b\,s-t)^2}\,ds\,dt \\
&= 4 + 2\int_{-1}^{1} (a+b\,s)\,\left[\log(-a-b\,s+1) - \log(-a-b\,s-1)\right] \\
&\qquad + (a+b\,s)^2\left[\frac{1}{a+b\,s-1} - \frac{1}{a+b\;+1}\right]\,ds.
\end{aligned}
$$

We proceed as in Lemma **??** and simplify the logarithm terms by point reflection. Thus, we get

$$
\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1} \frac{t^2}{(a+b\,s-t)^2}\,ds\,dt \\
&= 4 + 2\int_{-1}^{1} (a+b\,s)\,\left[\log\left(\eta_1\left(\frac{1-a}{b}-s\right)\right) - \log\left(\eta_1\left(\frac{-1-a}{b}-s\right)\right)\right] \\
&\qquad + \frac{(a+b\,s)^2}{b}\left[\frac{1}{\frac{1-a}{b}-s} - \frac{1}{\frac{-1-a}{b}-s}\right]\,ds \\
&= 2a\left[\widetilde{Q}_0^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_0^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right] + 2b\left[\widetilde{Q}_1^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_1^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right] \\
&\quad - \left(\frac{a^2}{b} + \frac{1}{3}\,b\right)\left[\widetilde{Q}_0^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_0^0\left(\frac{-1-a}{b}\right)\right] - 2a\left[\widetilde{Q}_1^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^0\left(\frac{-1-a}{b}\right)\right] \\
&\quad - \frac{2}{3}\,b\left[\widetilde{Q}_2^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_2^0\left(\frac{-1-a}{b}\right)\right] + 4.
\end{aligned}
$$

For the second integral in A.1 we get

$$
\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} \frac{1}{(a+b\,s-t)^2}\,ds\,dt &= \int_{-1}^{1} \frac{1}{(a+b\,s-1)} - \frac{1}{(a+b\,s+1)}\,ds \\
&= \frac{1}{b}\left[\widetilde{Q}_0^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_0^0\left(\frac{-1-a}{b}\right)\right]
\end{aligned}
$$

Putting the result together, we get the first identity

$$
\begin{aligned}
O^1_{0,3}(a,b) &= a\left[\widetilde{Q}_0^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_0^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right] + b\left[\widetilde{Q}_1^{-1}\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^{-1}\left(\frac{-1-a}{b}\right)\right] \\
&\quad - \frac{1}{2}\left(\frac{a^2-1}{b} - \frac{1}{3}b\right)\left[\widetilde{Q}_0^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_0^0\left(\frac{-1-a}{b}\right)\right] \\
&\quad - a\left[\widetilde{Q}_1^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^0\left(\frac{-1-a}{b}\right)\right] - \frac{1}{3}b\left[\widetilde{Q}_2^0\left(\frac{1-a}{b}\right) - \widetilde{Q}_2^0\left(\frac{-1-a}{b}\right)\right] + 2.
\end{aligned}
$$

□

For the second identity, we get

$$O^1_{1,3}(a,b) = \frac{1}{2}\left(\int_{-1}^{1}\int_{-1}^{1}\frac{s\,t^2}{(a+b\,s-t)^2}\,ds\,dt - \int_{-1}^{1}\int_{-1}^{1}\frac{s}{(a+b\,s-t)^2}\,ds\,dt\right). \quad \text{(A.2)}$$

By integrating similarly as above, we obtain

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1}\frac{s\,t^2}{(a+b\,s-t)^2}\,ds\,dt\\
&= 4 + 2\int_{-1}^{1} s\,(a+b\,s)\,\left[\log\left(\eta_1\left(\frac{1-a}{b}-s\right)\right) - \log\left(\eta_1\left(\frac{-1-a}{b}-s\right)\right)\right]\\
&\qquad + \frac{s\,(a+b\,s)^2}{b}\left[\frac{1}{\frac{1-a}{b}-s} - \frac{1}{\frac{-1-a}{b}-s}\right]\,ds\\
&= 2a\left[\widetilde{Q}_1^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_1^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right]\\
&\quad + 2b\left[\frac{2}{3}\widetilde{Q}_2^{-1}\left(\eta_1\frac{1-a}{b}\right) - \frac{2}{3}\widetilde{Q}_2^{-1}\left(\eta_1\frac{-1-a}{b}\right) + \frac{1}{3}\widetilde{Q}_0^{-1}\left(\eta_1\frac{1-a}{b}\right) - \frac{1}{3}\widetilde{Q}_0^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right]\\
&\quad - \left(\frac{a^2}{b} + \frac{1}{3}b\right)\left[\widetilde{Q}_1^{0}\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^{0}\left(\frac{-1-a}{b}\right)\right]\\
&- 2a\left[\frac{2}{3}\widetilde{Q}_2^{-1}\left(\eta_1\frac{1-a}{b}\right) - \frac{2}{3}\widetilde{Q}_2^{-1}\left(\eta_1\frac{-1-a}{b}\right) + \frac{1}{3}\widetilde{Q}_0^{-1}\left(\eta_1\frac{1-a}{b}\right) - \frac{1}{3}\widetilde{Q}_0^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right]\\
&\quad - \frac{2}{3}\,b\left[\frac{3}{5}\widetilde{Q}_3^{0}\left(\frac{1-a}{b}\right) - \frac{3}{5}\widetilde{Q}_3^{0}\left(\frac{-1-a}{b}\right) + \frac{2}{5}\widetilde{Q}_1^{0}\left(\frac{1-a}{b}\right) - \frac{2}{5}\widetilde{Q}_1^{0}\left(\frac{-1-a}{b}\right)\right] + 4.
\end{aligned}$$

where we exploited that

$$\begin{aligned}
s\,P_0(s) &= P_1(s)\\
s\,P_1(s) &= \frac{2}{3}\,P_2(s) + \frac{1}{2}\,P_0(s)\\
s\,P_2(s) &= \frac{3}{5}\,P_3(s) + \frac{2}{5}\,P_1(s).
\end{aligned}$$

For the second integral in A.2 we get

$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1}\frac{s}{(a+b\,s-t)^2}\,ds\,dt &= \int_{-1}^{1}\frac{s}{(a+b\,s-1)} - \frac{s}{(a+b\,s+1)}\,ds\\
&= \frac{1}{b}\left[\widetilde{Q}_1^{0}\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^{0}\left(\frac{-1-a}{b}\right)\right]
\end{aligned}$$

and finally we obtain

$$\begin{aligned}
O^1_{1,3}(a,b) &= a\left[\widetilde{Q}_1^{-1}\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^{-1}\left(\frac{-1-a}{b}\right)\right] + \frac{2}{3}b\left[\widetilde{Q}_2^{-1}\left(\frac{1-a}{b}\right) - \widetilde{Q}_2^{-1}\left(\frac{-1-a}{b}\right)\right]\\
&+ \frac{1}{3}b\left[\widetilde{Q}_0^{-1}\left(\eta_1\frac{1-a}{b}\right) - \widetilde{Q}_0^{-1}\left(\eta_1\frac{-1-a}{b}\right)\right]\\
&- \frac{1}{2}\left(\frac{a^2-1}{b} - \frac{3}{5}b\right)\left[\widetilde{Q}_1^{0}\left(\frac{1-a}{b}\right) - \widetilde{Q}_1^{0}\left(\frac{-1-a}{b}\right)\right]\\
&- \frac{2}{3}a\left[\widetilde{Q}_2^{0}\left(\frac{1-a}{b}\right) - \widetilde{Q}_2^{0}\left(\frac{-1-a}{b}\right)\right] - \frac{1}{3}a\left[\widetilde{Q}_0^{0}\left(\frac{1-a}{b}\right) - \widetilde{Q}_0^{0}\left(\frac{-1-a}{b}\right)\right]\\
&- \frac{1}{5}b\left[\widetilde{Q}_3^{0}\left(\frac{1-a}{b}\right) - \widetilde{Q}_3^{0}\left(\frac{-1-a}{b}\right)\right] + 2.
\end{aligned}$$

For proving the last identity we use two results that are stated in [**?**], i.e.

$$I^1_{j,1}(a,b) = b\,\frac{j}{2j+1}\,I^1_{j+1,0}(a,b) + b\,\frac{j+1}{2j+1}\,I^1_{j-1,0}(a,b) + a\,I^1_{j,0}(a,b)$$

and

$$I^1_{j,2}(a,b) = b\,\frac{j-1}{2j+1}\,I^1_{j+1,1}(a,b) + b\,\frac{j+2}{2j+1}\,I^1_{j-1,1}(a,b) + a\,I^1_{j,1}(a,b).$$

Combining both result yields

$$\begin{aligned}
I^1_{j,2}(a,b) = {} & ab\,\frac{3j}{(j+2)(2j+1)} I^1_{j+1,0}(a,b) + \left[a^2\,\frac{3}{j+2} + \frac{j-1}{j+2} + b^2\,\frac{3(j-1)}{(j+2)(2j-1)}\right] I^1_{j,0}(a,b) \\
& + ab\,\frac{3(3j+2)}{(j+2)(2j+1)} I^1_{j-1,0}(a,b) + b^2\,\frac{3j}{(j+2)(2j-1)} I^1_{j-2,0}(a,b).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
O^1_{j,3}(a,b) &= \frac{1}{3}\left(I^1_{j,2}(a,b) - I^1_{j,0}(a,b)\right) \\
&= ab\,\frac{j}{(j+2)(2j+1)} I^1_{j+1,0}(a,b) + \left[a^2\,\frac{1}{j+2} - \frac{1}{j+2} + b^2\,\frac{j-1}{(j+2)(2j-1)}\right] I^1_{j,0}(a,b) \\
&\quad + ab\,\frac{3j+2}{(j+2)(2j+1)} I^1_{j-1,0}(a,b) + b^2\,\frac{j}{(j+2)(2j-1)} I^1_{j-2,0}(a,b).
\end{aligned}$$

Plugging the initial values of $I^1_{j,0}(a,b)$, that are given by

$$I^1_{j,0}(a,b) = \frac{1}{b}\left[\widetilde{Q}^0_j\left(\frac{-1-a}{b}\right) - \widetilde{Q}^0_j\left(\frac{1-a}{b}\right)\right],$$

completes the proof.

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Ich bin mir bewusst, dass eine unwahre Erklärung rechtliche Folgen haben wird.

Ulm, den 29.08.2012

(Unterschrift)