Before we switch over to the construction of the a-posteriori error estimator $\Delta^N(\mu)$, we remark that there exist several extensions/modifications of Algorithm 4.4.1. In [18] the training set, i.e. $\Xi^{\text{train}} \subset \mathcal{D}_{\text{ad}}$, is no longer static but adaptively refined whenever the next parameter is identified, in [4] the greedy procedure itself is replaced by a steepest descent method in order to identify the next parameter value, such that high-dimensional spaces of parameters become better treatable, and in [13] an "hp" version has been introduced, which we are going to describe very briefly at the end of Section 4.5.1, where we are going to propose our extension.

## 4.5 A-Posteriori Error Estimation

In this section we finally derive the a-posteriori error estimator for the field variable, i.e. for

$$\varepsilon^N(\mu) := u(\mu) - \hat{u}^N(\mu), \tag{4.5.1}$$

where $u(\mu)$ solves $P(\mu)$ and $\hat{u}^N(\mu)$ is the solution of $\hat{P}^N(\mu)$. As pointed out in Section 4.4.2 it is important, that the a-posteriori error estimator to be derived is rapidly evaluable, rigorous and effective.

Since the presented a-posteriori error estimator is residual-based we start by defining

$$r^N_{\delta_1}(v;\mu) := f^{\delta_1}(v;\mu) - a^{\delta_1}(\hat{u}^N(\mu), v;\mu), \tag{4.5.2a}$$

$$\begin{aligned} r^N_{\delta_1,\delta_2}(v;\mu) &:= (r^N_{\delta_2} - r^N_{\delta_1})(v;\mu) \\ &= (f^{\delta_2} - f^{\delta_1})(v;\mu) - (a^{\delta_2} - a^{\delta_1})(u^N(\mu), v;\mu), \end{aligned} \tag{4.5.2b}$$

where $\delta_1 \geq \delta_2 \geq 0$, $v \in X$ and $\mu \in \mathcal{D}_{\text{ad}}$. Note, that we use the convention $a^0 := a$ and $f^0 := f$. Hence, $r^N_0(\mu)$ is the residual of $P(\mu)$ and $r^N_\epsilon(\mu)$ is the residual of $\hat{P}(\mu)$ both w.r.t. $\hat{u}^N(\mu)$. We continue by defining the corresponding Riesz representations, i.e. $\mathcal{R}^N_{\delta_1}(\mu), \mathcal{R}^N_{\delta_1,\delta_2}(\mu) \in X$, such that

$$(\mathcal{R}^N_{\delta_1}(\mu), v)_X = r^N_{\delta_1}(v;\mu), \qquad v \in X, \tag{4.5.3a}$$

$$(\mathcal{R}^N_{\delta_1,\delta_2}(\mu), v)_X = r^N_{\delta_1,\delta_2}(v;\mu), \quad v \in X, \tag{4.5.3b}$$

where from the Riesz representation theorem we infer that

$$R^N_{\delta_1}(\mu) := \|r^N_{\delta_1}(\cdot;\mu)\|_{X'} = \|\mathcal{R}^N_{\delta_1}(\mu)\|_X, \tag{4.5.4a}$$

$$R^N_{\delta_1,\delta_2}(\mu) := \|r^N_{\delta_1,\delta_2}(\cdot;\mu)\|_{X'} = \|\mathcal{R}^N_{\delta_1,\delta_2}(\mu)\|_X, \tag{4.5.4b}$$

where for $h \in X'$ we denote $\|h\|_{X'} := \sup_{v\in X} \frac{h(v)}{\|v\|_X}$.

Having this notation at hand, we can prove the following main result:

**Proposition 4.5.1.** *Let $\hat{\beta}(\mu)$ denote a positive lower bound for the inf-sup stability factor $\beta(\mu)$ (c.p. (4.1.1)), i.e. $0 < \hat{\beta}(\mu) \leq \beta(\mu)$. Moreover, let $E^N_\epsilon(\mu)$ denote an upper bound for $R^N_{\epsilon,0}(\mu)$,* *i.e.* $R^N_{\epsilon,0}(\mu) \leq E^N_\epsilon(\mu)$. *Then the a-posteriori error estimator*

$$\Delta^N_\epsilon(\mu) := \frac{1}{\hat{\beta}(\mu)}\big(R^N_\epsilon(\mu) + E^N_\epsilon(\mu)\big) \tag{4.5.5}$$

*is*

1. *rigorous, i.e. we have*

$$1 \leq \frac{\Delta^N_\epsilon(\mu)}{\|\varepsilon^N(\mu)\|_X} =: \eta^N_\epsilon(\mu), \tag{4.5.6}$$

2. *and effective, i.e. we have*

$$\eta^N_\epsilon(\mu) \leq \frac{1+c(\mu)}{1-c(\mu)}\frac{\gamma(\mu)}{\hat{\beta}(\mu)}, \tag{4.5.7}$$

*if there exists a constant* $c(\mu) \in [0,1)$, *such that*

$$E^N_\epsilon(\mu) \leq c(\mu) R^N_\epsilon(\mu). \tag{4.5.8}$$

*Proof.* For the first inequality, i.e. (4.5.6), by the bi-linearity of $a$, the definitions of $\varepsilon^N(\mu)$ and $P(\mu)$, and (4.5.2) we find

$$\begin{aligned} a(\varepsilon^N(\mu), v; \mu) &= a(u(\mu), v; \mu) - a(\hat{u}^N(\mu), v; \mu) \\ &= f(v;\mu) - a(\hat{u}^N(\mu), v; \mu) \\ &= r^N_0(v;\mu) \\ &= r^N_\epsilon(v;\mu) + r^N_{\epsilon,0}(v;\mu). \end{aligned} \tag{4.5.9}$$

Using this, the definition of $\Delta^N_\epsilon(\mu)$, the definition of the inf-sup stability factor $\beta(\mu)$ (c.p. (4.1.2)) and the fact that by assumption $E^N_\epsilon(\mu) \geq R^N_{\epsilon,0}(\mu)$, yields rigorousity, as

$$\begin{aligned} \hat{\beta}(\mu)\Delta^N_\epsilon(\mu) &= R^N_\epsilon(\mu) + E^N_\epsilon(\mu) \\ &\geq R^N_\epsilon(\mu) + R^N_{\epsilon,0}(\mu) \\ &\geq \|a(\varepsilon^N(\mu), \cdot; \mu)\|_{X'} \\ &\geq \beta(\mu)\|\varepsilon^N(\mu)\|_X, \end{aligned} \tag{4.5.10}$$

and by assumption $\hat{\beta}(\mu) \leq \beta(\mu)$, hence $\frac{\beta(\mu)}{\hat{\beta}(\mu)} \geq 1$.

Proving the effectivity of $\Delta^N_\epsilon(\mu)$, i.e. (4.5.7) is more involved. We start by observing that by

(4.5.8) we have

$$\begin{aligned}
&(1+c(\mu))\big(R_\epsilon^N(\mu)-E_\epsilon^N(\mu)\big)\\
&=(1-c(\mu))R_\epsilon^N(\mu)+2c(\mu)R_\epsilon^N(\mu)-(1+c(\mu))E_\epsilon^N(\mu)\\
&\geq(1-c(\mu))\big(R_\epsilon^N(\mu)+E_\epsilon^N(\mu)\big).
\end{aligned}\tag{4.5.11}$$

Next, using $v=\mathcal{R}_\epsilon^N(\mu)$ and $v=-\mathcal{R}_{\epsilon,0}^N(\mu)$, respectively, in (4.5.9) yields (recall (4.5.3))

$$\begin{aligned}
\big(\mathcal{R}_\epsilon^N(\mu),\mathcal{R}_\epsilon^N(\mu)\big)_X+\big(\mathcal{R}_{\epsilon,0}^N(\mu),\mathcal{R}_\epsilon^N(\mu)\big)_X&=a\big(\varepsilon^N(\mu),\mathcal{R}_\epsilon^N(\mu);\mu\big),\\
-\big(\mathcal{R}_\epsilon^N(\mu),\mathcal{R}_{\epsilon,0}^N(\mu)\big)_X-\big(\mathcal{R}_{\epsilon,0}^N(\mu),\mathcal{R}_{\epsilon,0}^N(\mu)\big)_X&=a\big(\varepsilon^N(\mu),-\mathcal{R}_{\epsilon,0}^N(\mu);\mu\big),
\end{aligned}$$

such that by summing up these two equations we obtain

$$\|\mathcal{R}_\epsilon^N(\mu)\|_X^2-\|\mathcal{R}_{\epsilon,0}^N(\mu)\|_X^2=a\big(\varepsilon^N(\mu),\mathcal{R}_\epsilon^N(\mu)-\mathcal{R}_{\epsilon,0}^N(\mu);\mu\big),\tag{4.5.12}$$

and using the continuity of $a$ (c.p. (4.1.1)) for the right hand side of (4.5.12) yields

$$a\big(\varepsilon^N(\mu),\mathcal{R}_\epsilon^N(\mu)-\mathcal{R}_{\epsilon,0}^N(\mu);\mu\big)\leq\gamma(\mu)\big(\|\mathcal{R}_\epsilon^N(\mu)\|_X+\|\mathcal{R}_{\epsilon,0}^N(\mu)\|_X\big)\|\varepsilon^N(\mu)\|_X.\tag{4.5.13}$$

Hence, from combining (4.5.12) and (4.5.13), using that by duality $\|\mathcal{R}_\epsilon^N(\mu)\|_X=R_\epsilon^N(\mu)$ and $\|\mathcal{R}_{\epsilon,0}^N(\mu)\|_X=R_{\epsilon,0}^N(\mu)$ (c.p. (4.5.4)) and the fact that $E_\epsilon^N(\mu)\geq R_{\epsilon,0}^N(\mu)$ by assumption, we obtain

$$R_\epsilon^N(\mu)-E_\epsilon^N(\mu)\leq R_\epsilon^N(\mu)-R_{\epsilon,0}^N(\mu)\leq\gamma(\mu)\|\varepsilon^N(\mu)\|_X.\tag{4.5.14}$$

Finally, using the definition of $\Delta_\epsilon^N(\mu)$, (4.5.11) and (4.5.14) yields the claim, as

$$\begin{aligned}
(1-c(\mu))\hat{\beta}(\mu)\Delta_\epsilon^N(\mu)&=(1-c(\mu))\big(R_\epsilon^N(\mu)+E_\epsilon^N(\mu)\big)\\
&\leq(1+c(\mu))\big(R_\epsilon^N(\mu)-E_\epsilon^N(\mu)\big)\\
&\leq(1+c(\mu))\gamma(\mu)\|\varepsilon^N(\mu)\|_X.
\end{aligned}$$

□

Next, before we discuss the implications of Proposition 4.5.1, we consider the case of $a$ to be symmetric and coercive for all $\mu\in\mathcal{D}_{\text{ad}}$, i.e.

$$\inf_{v\in X}\frac{a(v,v;\mu)}{\|v\|_X^2}=:\alpha(\mu)>0,\tag{4.5.15}$$

which allows for deriving a similar albeit sharper result than the one presented in Proposition 4.5.1, if the error $\varepsilon^N(\mu)$ is measured in the so-called energy norm $|\!|\!|\cdot|\!|\!|_\mu$ rather than in $\|\cdot\|_X$, where for $\mu \in \mathcal{D}_{\mathrm{ad}}$, $w, v \in X$ we define

$$(w,v)_\mu := a(w,v;\mu), \quad \|w\|_\mu := \sqrt{(w,w)_\mu}. \tag{4.5.16}$$

**Corollary 4.5.2.** *Let $\hat{\alpha}(\mu)$ denote a positive lower bound for the coercivity constant $\alpha(\mu)$ (c.p. (4.5.15)), i.e. $0 < \hat{\alpha}(\mu) \leq \alpha(\mu)$. Moreover, let $E_\epsilon^N(\mu)$ denote an upper bound for $R_{\epsilon,0}^N(\mu)$, i.e. $R_{\epsilon,0}^N(\mu) \leq E_\epsilon^N(\mu)$. Then the a-posteriori error estimator*

$$\Delta_{\mu,\epsilon}^N(\mu) := \frac{1}{\sqrt{\hat{\alpha}(\mu)}}\left(R_\epsilon^N(\mu) + E_\epsilon^N(\mu)\right) \tag{4.5.17}$$

*is*

1. *rigorous, i.e. we have*
$$1 \leq \frac{\Delta_{\mu,\epsilon}^N(\mu)}{\|\varepsilon^N(\mu)\|_\mu} =: \eta_{\mu,\epsilon}^N(\mu),$$
2. *and effective, i.e. we have*
$$\eta_{\mu,\epsilon}^N(\mu) \leq \frac{1+c(\mu)}{1-c(\mu)}\sqrt{\frac{\gamma(\mu)}{\hat{\alpha}(\mu)}}, \tag{4.5.18}$$
*if there exists a constant $c(\mu) \in [0,1)$, such that (4.5.8) holds true.*

*Proof.* The proof is very similar to the proof of Proposition 4.5.1. Using (4.5.9), the definitions of $\Delta_{\mu,\epsilon}^N(\mu)$, the energy norm $\|\cdot\|_\mu$ and the coercivity of $a$, i.e. (4.5.15), and the fact that by assumption $E_\epsilon^N(\mu) \geq R_{\epsilon,0}^N(\mu)$, yields rigorousity, as

$$\begin{aligned}
\sqrt{\hat{\alpha}(\mu)}\Delta_{\mu,\epsilon}^N(\mu) &= R_\epsilon^N(\mu) + E_\epsilon^N(\mu) \\
&\geq R_\epsilon^N(\mu) + R_{\epsilon,0}^N(\mu) \\
&\geq \|a(\varepsilon^N(\mu), \cdot; \mu)\|_{X'} \\
&\geq \frac{a(\varepsilon^N(\mu), \varepsilon^N(\mu); \mu)}{\|\varepsilon^N(\mu)\|_X} \\
&= \frac{a^{\frac{1}{2}}(\varepsilon^N(\mu), \varepsilon^N(\mu); \mu)}{\|\varepsilon^N(\mu)\|_X}\|\varepsilon^N(\mu); \mu)\|_\mu \\
&\geq \sqrt{\alpha(\mu)}\|\varepsilon^N(\mu); \mu)\|_\mu,
\end{aligned}$$

and by assumption $\hat{\alpha}(\mu) \leq \alpha(\mu)$, hence $\frac{\alpha(\mu)}{\hat{\alpha}(\mu)} \geq 1$.

For proving that $\Delta_{\mu,\epsilon}^N(\mu)$ is effective, we start with (4.5.12), where for the right hand side we apply the Cauchy-Schwarz inequality (recall that $(w,v)_\mu = a(w,v;\mu)$) in order to find

$$\begin{aligned}
\|\mathcal{R}_\epsilon^N(\mu)\|_X^2 - \|\mathcal{R}_{\epsilon,0}^N(\mu)\|_X^2 &\leq \|\mathcal{R}_\epsilon^N(\mu) - \mathcal{R}_{\epsilon,0}^N(\mu)\|_\mu \|\varepsilon^N(\mu)\|_\mu \\
&\leq \sqrt{\gamma(\mu)}\left(\|\mathcal{R}_\epsilon^N(\mu)\|_X + \|\mathcal{R}_{\epsilon,0}^N(\mu)(\mu)\|_X\right)\|\varepsilon^N(\mu)\|_\mu,
\end{aligned}$$

where we additionally used the continuity of $a$ (c.p. (4.1.1)) and the triangle inequality. Hence, again from duality and the fact that by assumption $E_\epsilon^N(\mu) \geq R_{\epsilon,0}^N(\mu)$, we obtain

$$R_\epsilon^N(\mu) - E_\epsilon^N(\mu) \leq R_\epsilon^N(\mu) - R_{\epsilon,0}^N(\mu) \leq \sqrt{\gamma(\mu)} \| \varepsilon^N(\mu) \|_\mu. \tag{4.5.19}$$

Finally, using the definition of $\Delta_{\mu,\epsilon}^N(\mu)$, (4.5.11) and (4.5.19) again yields the claim, as

$$\begin{aligned}(1 - c(\mu))\sqrt{\hat{\alpha}(\mu)}\Delta_{\mu,\epsilon}^N(\mu) &= (1 - c(\mu))\big(R_\epsilon^N(\mu) + E_\epsilon^N(\mu)\big) \\ &\leq (1 + c(\mu))\big(R_\epsilon^N(\mu) - E_\epsilon^N(\mu)\big) \\ &\leq (1 + c(\mu))\sqrt{\gamma(\mu)} \| \varepsilon^N(\mu) \|_\mu.\end{aligned}$$

□

The first thing to note is that if $a = a^\epsilon$ and $f = f^\epsilon$, i.e. if we are in the usual affine case, then $R_{\epsilon,0}^N(\mu)$ vanishes and for its upper bound we can choose $E_\epsilon^N(\mu) = 0$, which implies that (4.5.8) holds true for $c(\mu) = 0$, too, and the results reported in Proposition 4.5.1 and Corollary 4.5.2 coincide with the well-known results for the affine case. Hence, Proposition 4.5.1 and Corollary 4.5.2 are natural extensions to the standard theory. Comparing the results of Proposition 4.5.1 and Corollary 4.5.2 we find the factor of $\sqrt{\frac{\gamma(\mu)}{\hat{\alpha}(\mu)}}$ in (4.5.18) to be squared in (4.5.7), where we recall that in case of a symmetric and coercive form $a$ the values of coercivity constant and inf-sup stability factor coincide, i.e we have $\beta(\mu) = \alpha(\mu)$, which is also well-known. However, for our particular application $a$ is coercive, but not symmetric. Moreover, the problem derived in Section 4.6.2.1 is symmetric, but no longer coercive, such that we cannot take advantage of the sharper error bound provided by Corollary 4.5.2 at all and derived it mainly for the sake of completeness.

At the beginning of this section we formulated the goal of deriving an a-posteriori error estimator that is rapidly evaluable, rigorous and effective. At this point we have seen that the presented error estimators are rigorous, if $E_\epsilon^N(\mu)$ is an upper bound for $R_\epsilon^N(\mu)$, and effective, if $E_\epsilon^N(\mu)$ additionally satisfies (4.5.8), which leaves us with the postulation of rapid evaluability. Obviously, $\Delta_\epsilon^N(\mu)$ ($\Delta_{\mu,\epsilon}^N(\mu)$) is rapidly evaluable if and only if all three components it consists of, namely $\hat{\beta}(\mu)$ ($\hat{\alpha}(\mu)$), $R_\epsilon^N(\mu)$ and $E_\epsilon^N(\mu)$, are rapidly evaluable, such that we address each component separately in the sequel.

We start with the lower bound for the inf-sup stability factor and the coercivity constant, respectively, where we first note that in Corollary 4.5.2 we could readily use $\hat{\alpha}(\mu) = \alpha(\mu)$ and in Proposition 4.5.1 we could use $\hat{\beta}(\mu) = \beta(\mu)$ or even $\hat{\beta}(\mu) = \alpha(\mu)$ in the coercive case, as for the latter $0 < \alpha(\mu) \leq \beta(\mu)$ holds true. However, the computation of $\alpha(\mu)$ translates to solving a generalized eigenvalue problem, which in fact is more complex than solving the actual problem $P(\mu)$, and the computation of $\beta(\mu)$ is even more involved, which is why we substituted $\beta(\mu)$ and $\alpha(\mu)$ by lower bounds right from the start when formulating Proposition 4.5.1 and Corollary 4.5.2, respectively. Now, except for some special cases the derivation of such rapidly evaluable lower bounds is quite complex, such that at this point we assume these bounds to be given and postpone the details on the construction to Section 4.7, which leads us to the offline-/online decomposition of $R_\epsilon^N(\mu)$.

The offline-/online decomposition of the computation of $R_\epsilon^N(\mu)$ is quite standard. The key idea is to use the fact that by duality $R_\epsilon^N(\mu)$ can be computed via $\|\mathcal{R}_\epsilon^N(\mu)\|_X$ (c.p. (4.5.4)). Recall, that by (4.5.3a), (4.5.2a), (4.2.2) and the representation of $\hat{u}^N(\mu) \in X^N$, i.e. $\hat{u}^N(\mu) = \sum_{n=1}^N \hat{u}_n^N(\mu)\xi^n$, the Riesz representation $\mathcal{R}_\epsilon^N(\mu) \in X$ satisfies for $v \in X$

$$\begin{aligned}(\mathcal{R}_\epsilon^N(\mu), v)_X &= r_\epsilon^N(v;\mu)\\ &= f^\epsilon(v;\mu) - a^\epsilon(\hat{u}^N(\mu), v;\mu)\\ &= \sum_{m=1}^{M_f^\epsilon} \vartheta_m^f(\mu) f^m(v) - \sum_{m=1}^{M_a^\epsilon} \vartheta_m^a(\mu) \sum_{n=1}^{N} \hat{u}_n^N(\mu) a^m(\xi^n, v).\end{aligned}$$

Hence, from linear superposition we infer that $\mathcal{R}_\epsilon^N(\mu) \in X$ can be composed via

$$\mathcal{R}_\epsilon^N(\mu) = \sum_{m=1}^{M_f^\epsilon} \vartheta_m^f(\mu)\mathcal{F}^m + \sum_{m=1}^{M_a^\epsilon} \vartheta_m^a(\mu) \sum_{n=1}^{N} \hat{u}_n^N(\mu)\mathcal{A}_n^m, \tag{4.5.20}$$

where $\mathcal{F}^m, \mathcal{A}_n^m \in X$ satisfy for all $v \in X$

$$(\mathcal{F}^m, v)_X = f^m(v), \qquad 1 \le m \le M_\epsilon^f, \tag{4.5.21a}$$

$$(\mathcal{A}_n^m, v)_X = -a^m(\xi^n, v), \quad 1 \le m \le M_\epsilon^a,\ 1 \le n \le N. \tag{4.5.21b}$$

For notational convenience, we abbreviate (4.5.20) by

$$\mathcal{R}_\epsilon^N(\mu) = \sum_{m=1}^{M_c^\epsilon} \vartheta_m^c(\mu)\mathcal{C}^m,$$

where $M_c^\epsilon = M_f^\epsilon + M_a^\epsilon N$. Finally, we compute $R_\epsilon^N(\mu)$ from

$$(R_\epsilon^N(\mu))^2 = \|\mathcal{R}_\epsilon^N(\mu)\|_X^2 = \underline{\vartheta}^c(\mu) \cdot C\underline{\vartheta}^c(\mu),$$

where $\underline{\vartheta}^c(\mu) = (\vartheta_m^c(\mu))_{1\le m\le M_c^\epsilon} \in \mathbb{R}^{M_c^\epsilon}$ and the matrix $C \in \mathbb{R}^{M_c^\epsilon \times M_c^\epsilon}$ consists of the mutual inner products (w.r.t. $X$) of $\mathcal{C}^m$, $1 \le m \le M_c^\epsilon$, i.e.

$$C := \big((\mathcal{C}^m, \mathcal{C}^{m'})_X\big)_{1\le m,m'\le M_c^\epsilon}. \tag{4.5.22}$$

Hence, once we have the functions $\mathcal{C}^m \in X$, $1 \le m \le M_c^\epsilon$, and the matrix $C$, respectively, at hand, the (online) complexity for computing $R_\epsilon^N(\mu)$ is $\mathcal{O}((M_c^\epsilon)^2) = \mathcal{O}((M_f^\epsilon + M_a^\epsilon N)^2)$, which yields $\mathcal{N}$-independency. Obviously, in order to enable this online complexity, in the offline phase we have to compute $\mathcal{C}^m$, $1 \le m \le M_c^\epsilon$, which translates to solving $M_c^\epsilon = M_f^\epsilon + M_a^\epsilon N$ Poisson-like problems (c.p. (4.5.21)), which leaves us with the computation of $E_\epsilon^N(\mu)$.

The possibility of rapidly computing $E_\epsilon^N(\mu)$ obviously depends on its definition, as to this point we only required it to be an upper bound for $R^N_{\epsilon,0}(\mu)$. The best choice w.r.t. to the effectivity of the a-posterior error estimator would be to use $E^N_\epsilon(\mu) = R^N_{\epsilon,0}(\mu)$. However, as $r^N_{\epsilon,0}(\mu)$ involves (c.p. (4.5.2b)) the (possibly) non-affine forms $a$ and $f$, for its computation we cannot carry out an offline-/online decomposition as just presented for $R^N_\epsilon(\mu)$, hence we would loose $\mathcal{N}$-independency, which is why we stated Proposition 4.5.1 and Corollary 4.5.2 w.r.t. an upper bound for $R^N_{\epsilon,0}(\mu)$ right from the start. Now, instead of using $E^N_\epsilon(\mu) = R^N_{\epsilon,0}(\mu)$ we define for any $0 \le \delta_2 < \delta_1$:

$$E^N_{\delta_1,\delta_2}(\mu) := \begin{cases} \delta_1(1 + \|\hat{u}^N(\mu)\|_X), & \delta_2 = 0, \\ R^N_{\delta_1,\delta_2}(\mu) + E^N_{\delta_2,0}(\mu), & \delta_2 > 0. \end{cases} \tag{4.5.23}$$

Substituting $E^N_\epsilon(\mu)$ in (4.5.5) and (4.5.17) by $E^N_{\epsilon,\delta}(\mu)$ for any $0 \le \delta < \epsilon$ yields our final a-posteriori error estimators, i.e.

$$\Delta^N_{\epsilon,\delta}(\mu) := \frac{1}{\hat{\beta}(\mu)}\big(R^N_\epsilon(\mu) + E^N_{\epsilon,\delta}(\mu)\big), \tag{4.5.24a}$$

$$\Delta^N_{\mu,\epsilon,\delta}(\mu) := \frac{1}{\sqrt{\hat{\alpha}(\mu)}}\big(R^N_\epsilon(\mu) + E^N_{\epsilon,\delta}(\mu)\big). \tag{4.5.24b}$$

It is immediately obvious, that this substitution is indeed justified, as $E^N_{\epsilon,\delta}(\mu)$ is an upper bound for $R^N_{\epsilon,0}(\mu)$ for all $0 \le \delta < \epsilon$, as for $0 = \delta < \epsilon$ from (4.5.3b), (4.5.2b), (4.2.1) and (4.5.23) we infer that

$$\begin{aligned} R^N_{\epsilon,0}(\mu) &= \|r^N_{\epsilon,0}(\cdot;\mu)\|_{X'} \\ &= \|(f - f^\epsilon)(\cdot;\mu) - (a - a^\epsilon)(\hat{u}^N(\mu),\cdot;\mu)\|_{X'} \\ &\le \|(f - f^\epsilon)(\cdot;\mu)\|_{X'} + \|(a - a^\epsilon)(\hat{u}^N(\mu),\cdot;\mu)\|_{X'} \\ &\le \epsilon + \epsilon\|\hat{u}^N(\mu)\|_X \\ &= E^N_{\epsilon,\delta}(\mu), \end{aligned}$$

and for $0 < \delta < \epsilon$, we simply use

$$R^N_{\epsilon,0}(\mu) \le R^N_{\epsilon,\delta}(\mu) + R^N_{\delta,0}(\mu) \le R^N_{\epsilon,\delta}(\mu) + E^N_{\delta,0}(\mu) = E^N_{\epsilon,\delta}(\mu).$$

For the offline-/online decomposition of $E^N_{\epsilon,\delta}(\mu)$ we note that $\|\hat{u}^N(\mu)\|_X$ can be computed (online) in $\mathcal{O}(N)$, as the reduced-basis functions are normalized w.r.t. the inner product in $X$, hence $\|\hat{u}^N(\mu)\|_X = \|\underline{\hat{u}}^N(\mu)\|_2$. Moreover, for $0 < \delta < \epsilon$ the decomposition of $R^N_{\epsilon,\delta}(\mu)$ works along the lines of $R^N_\epsilon(\mu)$ presented above, such that the online complexity yields $\mathcal{O}\big(((M^\delta_f - M^\epsilon_f) + (M^\delta_a - M^\epsilon_a)N)^2\big)$, which finalizes the derivation of our rigorous, effective and rapidly evaluable (i.e. in a complexity independent of $\mathcal{N}$) a-posteriori error estimator.

Before we proceed we recall that in line four of Algorithm 4.4.1 we stated to "update all remaining offline quantities". Now, it is clear that this, e.g. for the computation of $R^N_\epsilon(\mu)$, also involves the solution of the Poisson-like problems (4.5.21b) for the current $N$ and $1 \le m \le M^\epsilon_a$