(a) $\underline{\underline{\breve{\alpha}}} = \underline{\underline{\breve{\alpha}}}_{\mathrm{diag}}$

$\Delta\nu$	$tol = 1e-1$		$tol = 1e-2$		$tol = 1e-3$		$tol = 1e-4$	
5°	1223	976	1160	852	1060	716	861	502
15°	1131	797	1002	623	781	399	568	255
25°	1064	686	878	479	583	262	356	141
35°	953	556	701	342	447	187	221	84

(b) $\underline{\underline{\breve{\alpha}}} = \underline{\underline{\breve{\alpha}}}_{\mathrm{sym}}$

$\Delta\nu$	$tol = 1e-1$		$tol = 1e-2$		$tol = 1e-3$		$tol = 1e-4$	
5°	1102	717	982	526	835	355	633	191
15°	929	462	748	268	475	122	292	58
25°	793	301	542	140	305	61	137	24
35°	671	198	409	87	177	31	82	14

Table 4.3: Computational savings for computing $\hat{\mathbf{s}}^N_{\mathscr{F},\zeta}(\mu)$ (first columns) as well as $\hat{\mathbf{s}}^N_{\mathscr{F},\zeta}(\mu)$ plus $\Delta^N_{\mathbf{s}_{\mathscr{F}},\zeta}(\mu)$ (second columns) in dependency on $\Delta\nu$ and tol, where $N = N(tol)$ and $\zeta = \zeta(tol)$ are chosen, such that $\Delta^N_{\mathbf{s}_{\mathscr{F}},\zeta}(\mu) \leq tol\, \hat{\mathbf{s}}^N_{b,0}(\mu)$ for all $\mu \in \Xi^{\mathrm{train}} \subset \mathcal{D}^{0^\circ,\Delta\nu}_{\mathrm{ad}}$.

comprises all remaining computations needed for evaluating $\Delta^N_{\mathbf{s}_{\mathscr{F}},\zeta}(\mu)$ (again except for the evaluation of $\hat{\beta}_{\mathscr{A}}(\mu)$).

Incited by these factors, we want to determine in the next subsection whether it is beneficial to pursue another way of treating quadratic outputs that works without extending $P(\mu)$ and $D(\mu)$ to one big problem.

4.6.2.2 Non-Extended Formulation

The second method for obtaining an improved a-posteriori error estimator for quadratic outputs is more intuitive, which is why it turned out to be the predecessor of the method presented in the last section and can be found e.g. in [23]. The reason for presenting the predecessor after the successor is quite simple. The method presented in the last section efficiently eliminates the quadratic non-linearity in the output by shifting the output bi-linear form b to the left hand side allowing us to tackle the transformed problem with what we already knew to that point, whereas the method to be presented next requires a new way of proceeding.

As for non-compliant linear outputs we start by introducing a dual problem, which reads as follows.

Definition 4.6.9. *Let $u(\mu) \in X$ be the solution of $P(\mu)$ and $\hat{u}^N(\mu) \in X^N$ be the solution of $\hat{P}^N(\mu)$.*

1. *Then, finding* $\psi^N(\mu) \in X$*, such that*

$$a(v, \psi^N(\mu); \mu) = b(u(\mu) + \hat{u}^N(\mu), v; \mu), \quad v \in X, \qquad (D^N(\mu))$$

is called Dual Problem *for the quadratic output* $\mathbf{s}_b(\mu)$*.*

2. *Let* $a^{\tilde{\epsilon}}$ *be as in Definition 4.2.1 and* $b^{\tilde{\epsilon}}$ *be as in Assumption 4.6.2. Moreover, let* $\tilde{X}^{\tilde{N}} := \operatorname{span}\{\tilde{\xi}^1, \ldots, \tilde{\xi}^{\tilde{N}}\} \subset X$*. Then, finding* $\hat{\psi}^{N,\tilde{N}}(\mu) \in \tilde{X}^{\tilde{N}}$*, such that*

$$a^{\tilde{\epsilon}}(v, \hat{\psi}^{N,\tilde{N}}(\mu); \mu) = b^{\tilde{\epsilon}}(2\hat{u}^N(\mu), v; \mu), \quad v \in \tilde{X}^{\tilde{N}}, \qquad (\hat{D}^{N,\tilde{N}}(\mu))$$

is called the corresponding Dual Reduced-Order Model.

Before we continue, we have to comment on this definition. In contrast to the adjoint equation from the last section, i.e. (4.6.25b), $D^N(\mu)$ not only depends on the solution of the primal problem $P(\mu)$, i.e. $u(\mu)$, but additionally on the solution of the primal reduced-order model $\hat{P}^N(\mu)$, i.e. $\hat{u}^N(\mu)$, which we will also see in Chapter 6 when dealing with quadratic non-linearities in the left hand side of the equation. Moreover, one obviously wants to avoid a build up of a reduced-basis approximation space for each $1 \leq N \leq N^{\max}$, which is why $\tilde{X}^{\tilde{N}}$ does not depend on N in the definition of $\hat{D}^{N,\tilde{N}}(\mu)$. In order to achieve this, let $\tilde{S}^{\tilde{N}} := \{\tilde{\mu}^1, \ldots, \tilde{\mu}^{\tilde{N}}\}$ denote the set of (identified) parameters as usual. Then, for $\tilde{X}^{\tilde{N}}$ one chooses the span of solutions of $D^{N^{\max}}(\mu)$ for all values in $\tilde{S}^{\tilde{N}}$, i.e.

$$\tilde{X}^{\tilde{N}} := \operatorname{span}\{\psi^{N^{\max}}(\tilde{\mu}^1), \ldots, \psi^{N^{\max}}(\tilde{\mu}^{\tilde{N}})\},$$

which also implies that the construction of the bases for $P(\mu)$ and $D^N(\mu)$ has to be performed one after another which at least could be avoided for non-compliant linear outputs (c.p. Section 4.6.1.2). Finally, while comparing $D^N(\mu)$ and $\hat{D}^{N,\tilde{N}}(\mu)$ one observes that in the first argument of b the term $\varepsilon^N(\mu)$ (c.p. (4.5.1)) has been left out, as

$$u(\mu) + \hat{u}^N(\mu) = 2\hat{u}^N(\mu) + \varepsilon^N(\mu), \qquad (4.6.32)$$

since it involves the solution of $P(\mu)$, i.e. $u(\mu)$. Hence, this omission has to be taken into account while developing an a-posteriori error estimator for

$$\tilde{\varepsilon}^{N,\tilde{N}}(\mu) := \psi^N(\mu) - \hat{\psi}^{N,\tilde{N}}(\mu).$$

In order to do so, we start again by defining in analogy to (4.5.2) and (4.5.4):

$$\tilde{r}^{N,\tilde{N}}_{\delta_1}(v;\mu) := b^{\delta_1}(2\hat{u}^N(\mu), v; \mu) - a^{\delta_1}(v, \hat{\psi}^{N,\tilde{N}}(\mu); \mu), \quad \tilde{R}^{N,\tilde{N}}_{\delta_1}(\mu) := \|\tilde{r}^{N,\tilde{N}}_{\delta_1}(\cdot;\mu)\|_{X'}, \qquad (4.6.33a)$$

$$\tilde{r}^{N,\tilde{N}}_{\delta_1,\delta_2}(v;\mu) := \tilde{r}^{N,\tilde{N}}_{\delta_2}(v;\mu) - \tilde{r}^{N,\tilde{N}}_{\delta_1}(v;\mu), \quad \tilde{R}^{N,\tilde{N}}_{\delta_1,\delta_2}(\mu) := \|\tilde{r}^{N,\tilde{N}}_{\delta_1,\delta_2}(\cdot;\mu)\|_{X'}, \qquad (4.6.33b)$$

where $\delta_1 \geq \delta_2 \geq 0$, $v \in X$ and $\mu \in \mathcal{D}_{\text{ad}}$, and in analogy to (4.5.23):

$$\tilde{E}^{N,\tilde{N}}_{\delta_1,\delta_2}(\mu) := \begin{cases} \delta_1(2\|\hat{u}^N(\mu)\|_X + \|\hat{\psi}^{N,\tilde{N}}(\mu)\|_X), & \delta_2 = 0, \\ \tilde{R}^{N,\tilde{N}}_{\delta_1,\delta_2}(\mu) + \tilde{E}^{N,\tilde{N}}_{\delta_2,0}(\mu), & \delta_2 > 0. \end{cases}$$

Having these quantities at hand, we can again present the a-posteriori error estimator for $\tilde{\varepsilon}^{N,\tilde{N}}(\mu)$ in the following proposition.

Proposition 4.6.10. *Let $\hat{\beta}(\mu)$ denote the same positive lower bound for the inf-sup stability factor $\beta(\mu)$ as in Proposition 4.5.1 and $\hat{\gamma}_b(\mu)$ an upper bound for $\gamma_b(\mu)$. Moreover, let $0 \leq \tilde{\delta} < \tilde{\epsilon}$. Then, the a-posteriori error estimator*

$$\tilde{\Delta}^{N,\tilde{N}}_{\tilde{\epsilon},\tilde{\delta}}(\mu) := \frac{1}{\hat{\beta}(\mu)}\big(\tilde{R}^{N,\tilde{N}}_{\tilde{\epsilon}}(\mu) + \tilde{E}^{N,\tilde{N}}_{\tilde{\epsilon},\tilde{\delta}}(\mu) + \hat{\gamma}_b(\mu)\Delta^N(\mu)\big)$$

for $\|\tilde{\varepsilon}^{N,\tilde{N}}(\mu)\|_X$ is

1. *rigorous, i.e. we have*

$$1 \leq \frac{\tilde{\Delta}^{N,\tilde{N}}_{\tilde{\epsilon},\tilde{\delta}}(\mu)}{\|\tilde{\varepsilon}^{N,\tilde{N}}(\mu)\|_X} =: \tilde{\eta}^{N,\tilde{N}}_{\tilde{\epsilon},\tilde{\delta}}(\mu),$$

2. *and effective, i.e. we have*

$$\tilde{\eta}^{N,\tilde{N}}_{\tilde{\epsilon},\tilde{\delta}}(\mu) \leq \frac{1+\tilde{c}(\mu)}{1-\tilde{c}(\mu)}\frac{\gamma(\mu)}{\hat{\beta}(\mu)}, \tag{4.6.34}$$

if there exists a constant $\tilde{c}(\mu) \in [0,1)$, such that

$$\tilde{E}^{N,\tilde{N}}_{\tilde{\epsilon},\tilde{\delta}}(\mu) + \hat{\gamma}_b(\mu)\Delta^N(\mu) \leq \tilde{c}(\mu)\tilde{R}^{N,\tilde{N}}_{\tilde{\epsilon}}(\mu). \tag{4.6.35}$$

Proof. Apart from some modifications the proof can be done along the lines of the proof for Proposition 4.5.1. For proving the rigorousity of $\tilde{\Delta}^{N,\tilde{N}}_{\tilde{\epsilon},\tilde{\delta}}(\mu)$ we start by observing that similar to (4.5.9) it holds true that

$$\begin{aligned} a(v,\tilde{\varepsilon}^{N,\tilde{N}}(\mu);\mu) &= a(v,\psi^N(\mu);\mu) - a(v;\hat{\psi}^{N,\tilde{N}}(\mu);\mu) \\ &= b(u(\mu)+\hat{u}^N(\mu),v;\mu) - a(v,\hat{\psi}^{N,\tilde{N}}(\mu);\mu) \\ &= b(2\hat{u}^N(\mu),v;\mu) - a(v,\hat{\psi}^{N,\tilde{N}}(\mu);\mu) + b(\varepsilon^N(\mu),v;\mu) \\ &= \tilde{r}^{N,\tilde{N}}_0(v;\mu) + b(\varepsilon^N(\mu),v;\mu) \\ &= \tilde{r}^{N,\tilde{N}}_{\tilde{\epsilon}}(v;\mu) + \tilde{r}^{N,\tilde{N}}_{\tilde{\epsilon},0}(v;\mu) + b(\varepsilon^N(\mu),v;\mu), \end{aligned} \tag{4.6.36}$$

where we have used the definition of $D^N(\mu)$, (4.6.32) and (4.6.33). From applying (4.6.8) for

(4.6.36) we obtain

$$\begin{aligned}\beta(\mu)\|\tilde{\varepsilon}^{N,\tilde{N}}(\mu)\|_X &\le \|a(\cdot,\tilde{\varepsilon}^{N,\tilde{N}}(\mu);\mu)\|_{X'}\\ &\le \tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu) + \tilde{R}_{\tilde{\epsilon},0}^{N,\tilde{N}}(\mu) + \hat{\gamma}_b(\mu)\|\varepsilon^N(\mu)\|_X,\end{aligned}$$

which already yields rigorousity, as $\|\varepsilon^N(\mu)\|_X \le \Delta^N(\mu)$ by Proposition 4.5.1, $\hat{\beta}(\mu) \le \beta(\mu)$ by assumption and $\tilde{R}_{\tilde{\epsilon},0}^{N,\tilde{N}}(\mu) \le \tilde{E}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$ by construction.

In order to keep the proof of (4.6.34) more compact we introduce the following abbreviation:

$$\tilde{E}(\mu) := \tilde{E}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu) + \hat{\gamma}_b(\mu)\Delta^N(\mu).$$

Then, in analogy to (4.5.11) from (4.6.35) we infer that

$$(1-\tilde{c}(\mu))\big(\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu) + \tilde{E}(\mu)\big) \le (1+\tilde{c}(\mu))\big(\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu) - \tilde{E}(\mu)\big). \tag{4.6.37}$$

Moreover, by the same arguments used in (4.5.12)-(4.5.14) we obtain

$$\begin{aligned}\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu) - \tilde{R}_{\tilde{\epsilon},0}^{N,\tilde{N}}(\mu) &\le \gamma(\mu)\|\tilde{\varepsilon}^{N,\tilde{N}}(\mu)\|_X + \hat{\gamma}_b(\mu)\|\varepsilon^N(\mu)\|_X\\ &\le \gamma(\mu)\|\tilde{\varepsilon}^{N,\tilde{N}}(\mu)\|_X + \hat{\gamma}_b(\mu)\Delta^N(\mu),\end{aligned}$$

where for the last inequality we have again used Proposition 4.5.1. Next, shifting $\hat{\gamma}_b(\mu)\Delta^N(\mu)$ to the other side of this inequality and recalling that $\tilde{R}_{\tilde{\epsilon},0}^{N,\tilde{N}}(\mu) \le \tilde{E}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$ yields

$$\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu) - \tilde{E}(\mu) \le \gamma(\mu)\|\tilde{\varepsilon}^{N,\tilde{N}}(\mu)\|_X. \tag{4.6.38}$$

Finally, using the definition of $\tilde{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$, (4.6.37) and (4.6.38) yields the claim, as

$$\begin{aligned}(1-\tilde{c}(\mu))\hat{\beta}(\mu)\tilde{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu) &= (1-\tilde{c}(\mu))\big(\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu) + \tilde{E}(\mu)\big)\\ &\le (1+\tilde{c}(\mu))\big(\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu) - \tilde{E}(\mu)\big)\\ &\le (1+\tilde{c}(\mu))\gamma(\mu)\|\tilde{\varepsilon}^{N,\tilde{N}}(\mu)\|_X.\end{aligned}$$

□

Before we are going to use this result for deriving an improved output approximation and a corresponding a-posteriori error estimator for $\mathbf{s}_b(\mu)$, we comment on the offline-/online decomposition as well as the (automatic) adjustment of the approximation and estimation tolerances $\tilde{\epsilon}$ and $\tilde{\delta}$. Starting with the first, we note that for fixed $\hat{u}^N(\mu)$ the right hand side of $\hat{D}^{N,\tilde{N}}(\mu)$ is a (parametric) linear form. Hence, the offline-/online decomposition for assembling $\hat{D}^{N,\tilde{N}}(\mu)$ works basically as described in Section 4.2, which yields an online complexity of $\mathcal{O}(M_b^{\tilde{\epsilon}}N\tilde{N} + M_a^{\tilde{\epsilon}}\tilde{N}^2)$. By (4.6.33a) the same argument also holds true for the offline-/online decomposition for enabling the rapid evaluation of $\tilde{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$. Hence, we can again proceed as described in Section 4.5, which yields online complexities of $\mathcal{O}\big((M_b^{\tilde{\epsilon}}N + M_a^{\tilde{\epsilon}}\tilde{N})^2\big)$ and $\mathcal{O}\big(((M_b^{\tilde{\delta}} - M_b^{\tilde{\epsilon}})N + (M_a^{\tilde{\delta}} - M_a^{\tilde{\epsilon}})\tilde{N})^2\big)$ for evaluating $\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu)$ and $\tilde{E}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$. Recalling that we have postponed the rapid evaluation of $\hat{\gamma}_b(\mu)$ to Section 4.7, we can turn to the (automatic) adjustment of the approximation and estimation tolerances $\tilde{\epsilon}$ and $\tilde{\delta}$. For this task the result we have obtained in Proposition 4.6.10 is somewhat unhandy if we want to proceed in the same way as described in Section 4.5.1, as the summand $\hat{\gamma}_b(\mu)\Delta^N(\mu)$ in (4.6.35) does neither depend on $\tilde{\epsilon}$ nor $\tilde{\delta}$. On the other hand, we can divide $\tilde{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$ up into

$$\tilde{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu) = \hat{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu) + \frac{\hat{\gamma}_b(\mu)}{\hat{\beta}(\mu)}\Delta^N(\mu), \tag{4.6.39}$$

whereas by Corollary 4.6.5

$$\hat{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu) := \frac{1}{\hat{\beta}(\mu)}\big(\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu) + \tilde{E}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)\big)$$

is a rigorous a-posteriori error estimator for $\|\psi^N(\mu) - \hat{\psi}^{N,\tilde{N}}(\mu)\|_X$ if we would have stated $D^N(\mu)$ right from the start w.r.t. $2u^N(\mu)$ rather than $u(\mu)+u^N(\mu)$. Moreover, in this context $\hat{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$ is effective, if there exists a constant $\tilde{c}(\mu) \in [0,1)$, such that

$$\tilde{E}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu) \leq \tilde{c}(\mu)\tilde{R}_{\tilde{\epsilon}}^{N,\tilde{N}}(\mu).$$

Consequently, for $\hat{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$ we can proceed as described in Section 4.5.1 in order to obtain approximation and estimation tolerance adapted to $\tilde{N}$ in an optimal way, such that in the sequel we abbreviate

$$\hat{\Delta}^{N,\tilde{N}}(\mu) := \hat{\Delta}^{N,\tilde{N}}_{\tilde{\epsilon}(N^{\max},\tilde{N}),\tilde{\delta}(N^{\max},\tilde{N})}(\mu), \tag{4.6.40a}$$

$$\tilde{\Delta}^{N,\tilde{N}}(\mu) := \hat{\Delta}^{N,\tilde{N}}(\mu) + \frac{\hat{\gamma}_b(\mu)}{\hat{\beta}(\mu)}\Delta^N(\mu), \tag{4.6.40b}$$

Note, that we have used $N^{\max}$ rather than N for the same reasons $\tilde{X}^{\tilde{N}}$ is spanned by solutions of $D^{N^{\max}}(\mu)$, namely that we want to avoid deriving a reduced-order model for each $1 \leq N \leq N^{\max}$ independently, which then would naturally lead to $\tilde{\epsilon}(N,\tilde{N})$ and $\tilde{\delta}(N,\tilde{N})$ in (4.6.40a), too. Moreover, there is even a stronger argument for dividing up $\tilde{\Delta}_{\tilde{\epsilon},\tilde{\delta}}^{N,\tilde{N}}(\mu)$, which, however, cannot be presented until we have investigated the improved output approximation and the corresponding a-posteriori error estimator for $\mathbf{s}_b(\mu)$, which we want to do next.

Lemma 4.6.11. *Let $\hat{u}^N(\mu)$ and $\hat{\psi}^{N,\tilde{N}}(\mu)$ denote the solutions of $\hat{P}^N(\mu)$ and $\hat{D}^{N,\tilde{N}}(\mu)$, and $\zeta \geq 0$. Then, the* improved output approximation *for $\mathbf{s}_b(\mu)$ takes the form*

$$\hat{\mathbf{s}}_{b,\zeta}^{N,\tilde{N}}(\mu) := \hat{\mathbf{s}}_{b,\zeta}^{N}(\mu) + r_\zeta^N(\hat{\psi}^{N,\tilde{N}}(\mu);\mu),$$

and

$$\begin{aligned}\Delta^{N,\tilde{N}}_{\mathbf{s}_b,\zeta}(\mu) := {}&\hat{\beta}(\mu)\Delta^N(\mu)\tilde{\Delta}^{N,\tilde{N}}(\mu)\\ &+\zeta\big(\|\hat{u}^N(\mu)\|_X^2+\|\hat{u}^N(\mu)\|_X\|\hat{\psi}^{N,\tilde{N}}(\mu)\|_X+\|\hat{\psi}^{N,\tilde{N}}(\mu)\|_X\big)\end{aligned} \tag{4.6.41}$$

is a rigorous a-posteriori error estimator for $|\mathbf{s}_b(\mu)-\hat{\mathbf{s}}^{N,\tilde{N}}_{b,\zeta}(\mu)|$.

Proof. For proving that $\Delta^{N,\tilde{N}}_{\mathbf{s}_b,\zeta}(\mu)$ is an a-posteriori error estimator for $|\mathbf{s}_b(\mu)-\hat{\mathbf{s}}^{N,\tilde{N}}_{b,\zeta}(\mu)|$ we infer from (4.6.21) and the definition of $D^N(\mu)$ that

$$\mathbf{s}_b(\mu)-\hat{\mathbf{s}}^{N}_{b,\zeta}(\mu)-(b-b^\zeta)(\hat{u}^N(\mu),\hat{u}^N(\mu);\mu)=a(\varepsilon^N(\mu),\psi^N(\mu);\mu).$$

Using this, the definition of $\hat{\mathbf{s}}^{N,\tilde{N}}_{b,\zeta}(\mu)$ and (4.5.2b) yields

$$\begin{aligned}&\mathbf{s}_b(\mu)-\hat{\mathbf{s}}^{N,\tilde{N}}_{b,\zeta}(\mu)-(b-b^\zeta)(\hat{u}^N(\mu),\hat{u}^N(\mu);\mu)-r^N_{\zeta,0}(\hat{\psi}^{N,\tilde{N}}(\mu);\mu)\\ &=\mathbf{s}_b(\mu)-\hat{\mathbf{s}}^{N}_{b,\zeta}(\mu)-(b-b^\zeta)(\hat{u}^N(\mu),\hat{u}^N(\mu);\mu)-r^N_{0}(\hat{\psi}^{N,\tilde{N}}(\mu);\mu)\\ &=a(\varepsilon^N(\mu),\psi^N(\mu);\mu)-a(\varepsilon^N(\mu),\hat{\psi}^{N,\tilde{N}}(\mu);\mu)\\ &=a(\varepsilon^N(\mu),\tilde{\varepsilon}^{N,\tilde{N}}(\mu);\mu).\end{aligned} \tag{4.6.42}$$

The remaining part works along the lines of the proof of Lemma 4.6.6. First, we use Proposition 4.6.10 and (4.5.10) in order to estimate the absolute value of the right hand side of (4.6.42), which yields the first part of the a-posteriori error estimator $\Delta^{N,\tilde{N}}_{\mathbf{s}_b,\zeta}(\mu)$, as

$$\begin{aligned}|a(\varepsilon^N(\mu),\tilde{\varepsilon}^{N,\tilde{N}}(\mu);\mu)|&\le\|a(\varepsilon^N(\mu),\cdot;\mu)\|_{X'}\|\tilde{\varepsilon}^{N,\tilde{N}}(\mu)\|_X\\ &\le\hat{\beta}(\mu)\Delta^N(\mu)\tilde{\Delta}^{N,\tilde{N}}(\mu).\end{aligned} \tag{4.6.43}$$

For the second part we use (4.2.1) and (4.6.3b), which yields

$$\begin{aligned}|r^N_{\zeta,0}(\hat{\psi}^{N,\tilde{N}}(\mu);\mu)|&\le|(f-f^\zeta)(\hat{\psi}^{N,\tilde{N}}(\mu);\mu)|+|(a-a^\zeta)(\hat{u}^N(\mu),\hat{\psi}^{N,\tilde{N}}(\mu);\mu)|\\ &\le\zeta\|\hat{\psi}^{N,\tilde{N}}(\mu)\|_X+\zeta\|\hat{u}^N(\mu)\|_X\|\hat{\psi}^{N,\tilde{N}}(\mu)\|_X,\\ |(b-b^\zeta)(\hat{u}^N(\mu),\hat{u}^N(\mu);\mu)|&\le\zeta\|\hat{u}^N(\mu)\|_X^2.\end{aligned} \tag{4.6.44}$$

Hence, the claim again follows from (4.6.42), the triangle inequality as well as (4.6.43) and (4.6.44). □

For the rapid evaluation of $\hat{\mathbf{s}}^{N}_{b,\zeta}(\mu)$ and $r^N_\zeta(\hat{\psi}^{N,\tilde{N}}(\mu);\mu)$ we just note that this has already been investigated in (4.6.22)-(4.6.24) and (4.6.13)-(4.6.14), which leaves us with the argument for dividing up $\tilde{\Delta}^{N,\tilde{N}}_{\tilde{\epsilon},\tilde{\delta}}(\mu)$ by (4.6.39). This, however, turns out to be quite natural if we think of a strategy similar to the one presented in Section 4.6.1.1 in order to guarantee a prescribed output approximation accuracy at least in Ξ^{train}, as

$$\hat{\beta}(\mu)\Delta^N(\mu)\tilde{\Delta}^{N,\tilde{N}}(\mu) = \hat{\gamma}_b(\mu)\big(\Delta^N(\mu)\big)^2 + \hat{\beta}(\mu)\Delta^N(\mu)\hat{\Delta}^{N,\tilde{N}}(\mu). \tag{4.6.45}$$

Hence, we can proceed as follows:

1. Specify $tol > 0$, $\Xi^{\mathrm{train}} \subset \mathcal{D}_{\mathrm{ad}}$ and some weights $\omega_1(0.6), \omega_2(0.3) > 0$, such that $\omega_1 + \omega_2 < 1$.
2. Stop Algorithm 4.5.1 for $P(\mu)$ as soon as

$$\max_{\mu\in\Xi^{\mathrm{train}}} \hat{\gamma}_b(\mu)\big(\Delta^N(\mu)\big)^2 \leq \omega_1\, tol.$$

3. Stop Algorithm 4.5.1 for $D^N(\mu)$ as soon as

$$\max_{\mu\in\Xi^{\mathrm{train}}} \hat{\beta}(\mu)\Delta^N(\mu)\hat{\Delta}^{N,\tilde{N}}(\mu) \leq \omega_2\, tol.$$

4. Choose ζ, such that

$$\zeta \max_{\mu\in\Xi^{\mathrm{train}}} \big(\|\hat{u}^N(\mu)\|_X^2 + \|\hat{u}^N(\mu)\|_X \|\hat{\psi}^{N,\tilde{N}}(\mu)\|_X + \|\hat{\psi}^{N,\tilde{N}}(\mu)\|_X\big) \leq (1 - \omega_1 - \omega_2)\, tol.$$

Note that we can tackle relative approximation accuracies as well, again by dividing (4.6.41) and (4.6.45), respectively, by a low accurate approximation of $|\mathbf{s}_b(\mu)|$, e.g. $|\hat{\mathbf{s}}^N_{b,\delta(N)}(\mu)|$ is a reasonable choice.

Numerical Experiments†

In this paragraph we again perform several numerical experiments using the approach presented above for our particular application at hand, i.e. for approximating the quadratic output derived from the output bi-linear form b (c.p. (4.6.30)). Moreover, at the end of this paragraph we want to pick that method for dealing with quadratic outputs which is more advantageous in our situation.

As in the previous paragraphs of numerical experiments we start by presenting in Figure 4.24 the minimum number of reduced-basis functions for $\hat{P}^N(\mu)$ as well as $\hat{D}^{N,\tilde{N}}(\mu)$, such that

$$\hat{\gamma}_b(\mu)\big(\Delta^N(\mu)\big)^2 \leq \omega_1\, tol\, \hat{\mathbf{s}}^N_{b,0}(\mu) \quad \text{and} \quad \hat{\beta}(\mu)\Delta^N(\mu)\hat{\Delta}^{N,\tilde{N}}(\mu) \leq \omega_2\, tol\, \hat{\mathbf{s}}^N_{b,0}(\mu) \tag{4.6.46}$$

holds true for all $\mu \in \Xi^{\mathrm{train}}$. The basic observation is that $N(tol)$ is much larger than $\tilde{N}(tol)$, which, however, is not surprising considering the factors in front of the (products of the) a-posteriori error estimator for the field(s) as $\hat{\gamma}_b(\mu)$ takes much larger values than $\hat{\beta}(\mu)$. Moreover, we have to comment on the way these numbers are actually obtained, as this is the first time we deal with a dual problem, i.e. $D^N(\mu)$, depending on both $u(\mu)$, i.e. the solution of $P(\mu)$,

†All numerical experiments have been performed using MATLAB® R2007a and COMSOL Multiphysics® 3.4.