Prof. Dr. Stefan Funken
M.Sc. Mladjan Radic, Stefan Hain
Department of Numerical Mathematics
Ulm University

Numerik von gewöhnlichen Differenzialgleichungen
SoSe 2016

Sheet 7

Due June 9, 2016.

Exercise 1 (Stability Domains for One-Step Methods)

A method is called *A-stable*, if $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ is a subset of the domain in which the method is absolutely stable. This domain is called the *stability domain* of the method. To get this stability domain one proceeds as follows: For any method, applied to the *test problem*

$$\begin{aligned} y'(t) &= \lambda y(t), \qquad t \in [0,T], \\ y(0) &= 1 \end{aligned}$$

with $\lambda \in \mathbb{C}$, we consider the *stability polynomial* $\pi : \mathbb{C} \to \mathbb{C}$, defined by

$$\pi(z) = \rho(z) - h\lambda\sigma(z),$$

where $h > 0$ denotes the step-size of the method and ρ, σ are polynomials of degree n and m, respectively. Denoting by $z_j \in \mathbb{C}$ the roots of the stability polynomial, the stability domain is defined by

$$D_\lambda := \{z = h\lambda \in \mathbb{C} : |z_j(h\lambda)| < 1\}. \tag{1}$$

In the following the want to investigate the stability domains for one-step methods. As one can verify (1) is equivalent to

$$\mathcal{S} := \{z = h\lambda \in \mathbb{C} : |R(z)| < 1\},$$

where R is called the *stability function*, which is of the form $R(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials of degree n and m, respectively. One gets the stability function either by solving the equation

$$\pi(z) = \rho(z) - h\lambda\sigma(z) = 0$$

for z or by using any one-step method applied to the test problem (considered above), which yields

$$y_{k+1} = R(z)y_k = \ldots = R(z)^k y_0.$$

(a) Assume, that the polynomial Q has no roots in $\mathbb{C}_-$. Explain why

$$|R(iy)| \leq 1 \quad \text{for all } y \in \mathbb{R} \qquad \text{and} \qquad |R(\infty)| \leq 1$$

yields the A-stability of the associated method, e.g. $\mathbb{C}_- \subset \mathcal{S}$.

Consider for $\theta \in [0,1]$ the one-step method

$$y_{k+1} = y_k + h[(1-\theta)f(t_k, y_k) + \theta f(t_{k+1}, y_{k+1})].$$

For $\theta = 0$, $\theta = 0.5$ and $\theta = 1$ we get the following special cases:

- $\theta = 0$: explicit Euler method,
- $\theta = 0.5$: Crank-Nicolson method and

- $\theta = 1$: implicit Euler method.

(b) Determine the stability function R in dependence of θ.

(c) Create a plot (with Matlab), which shows the stability domains for various $\theta \in [0,1]$ in a common plot. For which values of θ is the method A-stable?

(d) Consider for $\lambda = 2000$ the IVP

$$\begin{aligned} y'(t) &= -\lambda\left(y(t) - \cos(t)\right), \qquad t \in [0,2], \\ y(0) &= 0, \end{aligned}$$

with the analytical solution $y(t) = -\frac{e^{-\lambda t}\lambda^2}{\lambda^2+1} + \frac{\lambda(\lambda\,\cos(t)+\sin(t))}{\lambda^2+1}$. Solve this problem with the implicit Euler method (with step-size $h = 0.1$, e.g. $N = 20$) and the Crank-Nicolson rule (with step-size $h = 0.1$ and $h = 0.05$, e.g. $N = 20$ and $N = 40$, respectively). Discuss and explain your results.

Exercise 2 (Stability Domain for explicit Runge-Kutta-Methods)

Let $f \in C^p(I \times \mathbb{R})$. Show, that all m-stepped explicit Runge-Kutta method of order $p = m$ has the stability function

$$R(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^m}{m!}.$$

Create a plot, which shows the stability domains for $m = 1, 2, 3, 4, 5$ (with Matlab) in a common plot. Justify, by using this task, why explicit Runge-Kutta methods are not A-stable.

Exercise 3 (Stiff ODE-System)

Consider the IVP

$$\begin{aligned} y_1'(t) &= -0.1y_1(t) + 100y_2(t)y_3(t), \\ y_2'(t) &= 0.1y_1(t) - 100y_2(t)y_3(t) - 500y_2(t)^2, \\ y_3'(t) &= 500y_2(t)^2 - 0.5y_3(t), \end{aligned} \tag{2}$$

for $t \in I := [0,25]$ with the initial values $y_1(0) = 4$, $y_2(0) = 2$ and $y_3(0) = 0.5$. This system describes the kinetics of a chemical reaction with the species Y_1, Y_2 and Y_3 by the law of mass action, where $y_1(t)$, $y_2(t)$ and $y_3(t)$ denotes the corresponding concentrations at t.

(a) Determine the eigenvalues of the Jacobi-Matrix J_f of the ODE-System (2) at $t = 0$ and calculate

$$S := \frac{\max_{\mathrm{Re}(\lambda_j)<0} |\mathrm{Re}(\lambda_j)|}{\min_{\mathrm{Re}(\lambda_j)<0} |\mathrm{Re}(\lambda_j)|},$$

where λ_j denotes the j-th eigenvalue of J_f.

(b) Solve the IVP with a suitable method and justify your choice. Create a plot with your solutions.