Exercise 1 (Stability Domains for One-Step Methods)

A method is called \(A\)-stable, if \(\mathcal{C}^- := \{z \in \mathbb{C} : \text{Re}(z) < 0\}\) is a subset of the domain in which the method is absolutely stable. This domain is called the stability domain of the method. To get this stability domain one proceeds as follows: For any method, applied to the test problem

\[
y'(t) = \lambda y(t), \quad t \in [0, T],
y(0) = 1
\]

with \(\lambda \in \mathbb{C}\), we consider the stability polynomial \(\pi : \mathbb{C} \to \mathbb{C}\), defined by

\[
\pi(z) = \rho(z) - h\lambda \sigma(z),
\]

where \(h > 0\) denotes the step-size of the method and \(\rho, \sigma\) are polynomials of degree \(n\) and \(m\), respectively. Denoting by \(z_j \in \mathbb{C}\) the roots of the stability polynomial, the stability domain is defined by

\[
D_\lambda := \{z = h\lambda \in \mathbb{C} : |z_j(h\lambda)| < 1\}.
\]  \(\text{(1)}\)

In the following want to investigate the stability domains for one-step methods. As one can verify \(\text{(1)}\) is equivalent to

\[
S := \{z = h\lambda \in \mathbb{C} : |R(z)| < 1\},
\]

where \(R\) is called the stability function, which is of the form \(R(z) = \frac{P(z)}{Q(z)}\), where \(P\) and \(Q\) are polynomials of degree \(n\) and \(m\), respectively. One gets the stability function either by solving the equation

\[
\pi(z) = \rho(z) - h\lambda \sigma(z) = 0
\]

for \(z\) or by using any one-step method applied to the test problem (considered above), which yields

\[
y_{k+1} = R(z)y_k = \ldots = R(z)^ky_0.
\]

(a) Assume, that the polynomial \(Q\) has no roots in \(\mathcal{C}^-\). Explain why

\[
|R(iy)| \leq 1 \quad \text{for all } y \in \mathbb{R} \quad \text{and} \quad |R(\infty)| \leq 1
\]

yields the \(A\)-stability of the associated method, e.g. \(\mathcal{C}^- \subset S\).

Consider for \(\theta \in [0, 1]\) the one-step method

\[
y_{k+1} = y_k + h[(1 - \theta)f(t_k, y_k) + \theta f(t_{k+1}, y_{k+1})].
\]

For \(\theta = 0, \theta = 0.5\) and \(\theta = 1\) we get the following special cases:

- \(\theta = 0\): explicit Euler method,
- \(\theta = 0.5\): Crank-Nicolson method and
• $\theta = 1$: implicit Euler method.

(b) Determine the stability function $R$ in dependence of $\theta$.

(c) Create a plot (with Matlab), which shows the stability domains for various $\theta \in [0,1]$ in a common plot. For which values of $\theta$ is the method A-stable?

(d) Consider for $\lambda = 2000$ the IVP

\[
y'(t) = -\lambda (y(t) - \cos(t)), \quad t \in [0,2], \\
y(0) = 0,
\]

with the analytical solution $y(t) = -\frac{e^{-\lambda t} - \lambda t}{\lambda^2 + 1} + \frac{\lambda \cos(t) + \sin(t)}{\lambda^2 + 1}$. Solve this problem with the implicit Euler method (with step-size $h = 0.1$, e.g. $N = 20$) and the Crank-Nicolson rule (with step-size $h = 0.1$ and $h = 0.05$, e.g. $N = 20$ and $N = 40$, respectively). Discuss and explain your results.

Exercise 2 (Stability Domain for explicit Runge-Kutta-Methods)

Let $f \in C^p(I \times \mathbb{R})$. Show, that all $m$-stepped explicit Runge-Kutta method of order $p = m$ has the stability function

\[ R(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^m}{m!}. \]

Create a plot, which shows the stability domains for $m = 1, 2, 3, 4, 5$ (with Matlab) in a common plot. Justify, by using this task, why explicit Runge-Kutta methods are not A-stable.

Exercise 3 (Stiff ODE-System)

Consider the IVP

\[
\begin{align*}
y_1'(t) &= -0.1 y_1(t) + 100 y_2(t) y_3(t), \\
y_2'(t) &= 0.1 y_1(t) - 100 y_2(t) y_3(t) - 500 y_2(t)^2, \\
y_3'(t) &= 500 y_2(t)^2 - 0.5 y_3(t),
\end{align*}
\]

for $t \in I := [0,25]$ with the initial values $y_1(0) = 4$, $y_2(0) = 2$ and $y_3(0) = 0.5$. This system describes the kinetics of a chemical reaction with the species $Y_1$, $Y_2$ and $Y_3$ by the law of mass action, where $y_1(t)$, $y_2(t)$ and $y_3(t)$ denotes the corresponding concentrations at $t$.

(a) Determine the eigenvalues of the Jacobi-Matrix $J_f$ of the ODE-System (2) at $t = 0$ and calculate

\[ S := \max_{\text{Re}(\lambda_j) < 0} |\text{Re}(\lambda_j)| \frac{|\text{Re}(\lambda_j)|}{\min_{\text{Re}(\lambda_j) < 0} |\text{Re}(\lambda_j)|}, \]

where $\lambda_j$ denotes the $j$-th eigenvalue of $J_f$.

(b) Solve the IVP with a suitable method and justify your choice. Create a plot with your solutions.