Reduced Basis Exact Error Estimates with Wavelets

Mazen Ali[1] and Karsten Urban[1]

Ulm University, Institute of Numerical Mathematics, Helmholtzstr. 20, D-89081 Ulm, Germany, {mazen.ali@uni-ulm.de, karsten.urban@uni-ulm.de}

Abstract. A (multi-)wavelet expansion is used to derive a rigorous bound for the (dual) norm Reduced Basis residual. We show theoretically and numerically that the error estimator is online efficient, reliable and rigorous. It allows to control the exact error (not only with respect to a "truth" discretization).

1 Introduction

The Reduced Basis Method (RBM) is a widely used mathematical framework for model reduction of parameterized partial differential equations (PPDEs). We refer to [4,6] for recent books. One possible criticism of the RBM is that the reduction is based upon an a priorily fixed, so called "truth" discretization, which is assumed to be sufficiently fine in order to resolve the desired solution sufficiently well for all possible parameters. This means that an RB approximation can only be as good as the underlying truth. If this truth is not so "true", the RBM cannot be expected to yield a good approximation of the exact solution.

Thus, one would like to have an error estimator for the RB approximation with respect to the (unknown and not computable) exact solution of the PPDE. In this paper, we build upon a recent preprint [1], where we propose to use adaptive computations for the construction of the reduced model. In the present paper, we use such an adaptive method build upon (multi-)wavelets to construct an error estimator for the exact error. We show that this estimator is computable online efficient and gives sharp estimates (the latter statement is shown by numerical experiments).

In Section 2, we review the error estimates based upon the dual norm of the residual within the RBM, Section 3 is devoted to the wavelet-based error estimator and our numerical results are shown in Section 4.

2 Reduced Basis Method (RBM) Error Estimation

In order to shorten notation, we consider parametric (for a parameter μ in a parameter set $\mathcal{D} \subset \mathbb{R}^P$) elliptic boundary value problems on $\mathcal{X} := H_0^1(\Omega)$,

i.e., we look for $u(\mu) \in \mathcal{X}$ such that[1]

$$a(u(\mu), v; \mu) = \langle f(\mu), v \rangle \qquad \forall v \in \mathcal{X}, \tag{1}$$

where $f(\mu) \in \mathcal{X}' = H^{-1}(\Omega)$ is given and $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $\mathcal{X}'$ and $\mathcal{X}$ with pivot space $L_2(\Omega)$. The bilinear form $a(\cdot, \cdot; \mu)$ is assumed to be symmetric, coercive and bounded with constants $\alpha(\mu)$ and $\gamma(\mu)$, respectively.

2.1 Residual-Based Error Estimators

Typically, RB error estimates are *residual-based*, where the *residual* $r(w; \mu) \in \mathcal{X}'$, $\mu \in \mathcal{D}$, is defined for given $w \in \mathcal{X}$ by

$$\langle r(w; \mu), v \rangle := \langle f(\mu), v \rangle - a(w, v; \mu), \qquad v \in \mathcal{X}. \tag{2}$$

The equivalence of error and residual is straightforward and well-known

$$\alpha(\mu) \, \|u(\mu) - w\|_{\mathcal{X}} \leq \|r(w; \mu)\|_{\mathcal{X}'} \leq \gamma(\mu) \, \|u(\mu) - w\|_{\mathcal{X}}, \quad w \in \mathcal{X}. \tag{3}$$

If an approximation space $X_N \subset \mathcal{X}$ of small dimension $N \in \mathbb{N}$ is constructed and a (Galerkin) approximation $u_N(\mu) \in X_N$ for a given parameter value $\mu \in \mathcal{D}$ has been computed, we set

$$R_N(\mu) := \|r(u_N(\mu); \mu)\|_{\mathcal{X}'} = \sup_{v \in \mathcal{X}} \frac{\langle r(u_N(\mu); \mu), v \rangle}{\|u_N(\mu)\|_{\mathcal{X}}}, \tag{4}$$

i.e., the *dual* norm of the residual. Usually, this dual norm is not computable, in particular since the supremum in (4) is taken over the *infinite*-dimensional space $\mathcal{X}$. Based upon $R_N(\mu)$, the error estimator for the *exact error* $u(\mu) - u_N(\mu)$ reads

$$\|u(\mu) - u_N(\mu)\|_{\mathcal{X}} \leq \frac{1}{\alpha(\mu)} R_N(\mu) =: \Delta_N(\mu). \tag{5}$$

Hence, one also needs the coercivity constant $\alpha(\mu)$, e.g. by the Successive Constraint Method (SCM), [5], which, however, is not the topic of this paper.

2.2 (Theoretical) Computing via Affine Decomposition and Riesz Representation

Any known computational procedure for the computation of $R_N(\mu)$ is based upon the assumption that the bilinear form $a(\cdot, \cdot; \mu)$ and the right-hand side allow for a separation of parameters and variables (often -a bit misleading- called an *affine decomposition* in the parameter), i.e.,

$$a(w, v; \mu) = \sum_{q=1}^{Q_a} \vartheta_q^a(\mu) \, a_q(w, v), \qquad f(\mu) = \sum_{q=1}^{Q_f} \vartheta_q^f(\mu) \, f_q, \tag{6}$$

[1] We would like to stress the fact that all what is said here also holds for Petrov-Galerkin inf-sup-stable problems with different trial and test spaces.

where ϑ_q^a, $\vartheta_q^f : \mathcal{D} \to \mathbb{R}$ and $a_q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ as well as $f_q \in \mathcal{X}'$ are parameter-independent. Then, it is straightforward to show that also the residual is affine in the parameter, i.e., with parameter-independent $r_q : \mathcal{X} \to \mathcal{X}'$, we have

$$r(w;\mu) = \sum_{q=1}^{Q_r} \vartheta_q^r(\mu)\, r_q(w), \qquad w \in \mathcal{X}. \tag{7}$$

Keeping in mind that $\mathcal{X}'$ is a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{X}'}$, one can try to compute $R_N(\mu)$ by using the expansion of the RB-solution in terms of the basis functions $\{\xi_1, \ldots, \xi_N\}$ (to be specified later) of X_N, i.e.,

$$u_N(\mu) = \sum_{i=1}^{N} c_i(\mu)\, \xi_i, \qquad c_i(\mu) \in \mathbb{R}, \tag{8}$$

as follows[2]

$$\begin{aligned} R_N(\mu)^2 &= \|r(u_N(\mu);\mu)\|_{\mathcal{X}'}^2 = \big(r(u_N(\mu);\mu), r(u_N(\mu);\mu)\big)_{\mathcal{X}'} \\ &= \sum_{q,q'=1}^{Q_r} \sum_{i,i'=1}^{N} \vartheta_q^r(\mu)\, \vartheta_{q'}^r(\mu)\, c_i(\mu)\, c_{i'}(\mu) \big(r_q(\xi_i), r_{q'}(\xi_{i'})\big)_{\mathcal{X}'}. \end{aligned} \tag{9}$$

Obviously, the inner products

$$\boldsymbol{R}_{(q,i),(q',i')} := \big(r_q(\xi_i), r_{q'}(\xi_{i'})\big)_{\mathcal{X}'} \tag{10}$$

are parameter-independent. If these values can be precomputed (in an offline phase), (9) amounts $(Q_r N)^2$ terms so that $R_N(\mu)$ can be computed with complexity $\mathcal{O}(N^2)$ – independent of $\mathcal{N}$, which is called *online efficient*.

A possible way to compute the terms in (10) is by determining the *Riesz representations* $\hat{r}_{q,i} \in \mathcal{X}$ of $r_q(\xi_i) \in \mathcal{X}'$, that are given by

$$\big(\hat{r}_{q,i}, v\big)_{\mathcal{X}} = \langle r_q(\xi_i), v\rangle \qquad \forall v \in \mathcal{X}, \tag{11}$$

where $(\cdot,\cdot)_{\mathcal{X}}$ denotes the inner product in $\mathcal{X}$. Once these Riesz representations are determined, one can compute (10) by $\boldsymbol{R}_{(q,i),(q',i')} = \big(\hat{r}_{q,i}, \hat{r}_{q',i'}\big)_{\mathcal{X}}$. Doing so, one avoids the dual inner product. Moreover, all computations to determine $\boldsymbol{R}_{(q,i),(q',i')}$ can be done offline. One only has to store $(Q_r N)^2$ numbers which are combined with the parameter-dependent terms in (9).

However, this approach is only theoretically feasible since the computation of the Riesz representations in (11) would amount solving an *infinite-dimensional* problem.

[2] Note, that (9) amounts to take the square root, which is not a problem in terms of efficiency, but it is an issue with respect to accuracy – the well-known so-called "square root effect".

2.3 The "Truth"

In a standard RBM, the way-out is through a common detailed discretization $\mathcal{X}^{\mathcal{N}} \subset \mathcal{X}$, where $\mathcal{N} \in \mathbb{N}$ is the dimension of the "truth" space, which is assumed to be rich enough to resolve the unknown $u(\mu)$ sufficiently accurate for all parameters $\mu \in \mathcal{D}$, i.e., the error $\|u(\mu) - u^{\mathcal{N}}(\mu)\|_{\mathcal{X}}$ is sufficiently small, where $u^{\mathcal{N}}(\mu) \in \mathcal{X}^{\mathcal{N}}$ is the corresponding truth approximation. This detailed space $\mathcal{X}^{\mathcal{N}}$ is used in the offline phase to determine the reduced model by computing the snapshots

$$\xi_i := u^{\mathcal{N}}(\mu_i), \qquad \mu_i \in \mathcal{D},\ 1 \leq i \leq N,$$

and setting $X_N := \operatorname{span}\{\xi_i : 1 \leq i \leq N\}$. The choice of the snapshot samples μ_i is also based upon an error estimator of the form (4), but restricted to the detailed space, i.e.,

$$R_N^{\mathcal{N}}(\mu) := \|r(u_N(\mu);\mu)\|_{(\mathcal{X}^{\mathcal{N}})'} = \sup_{v^{\mathcal{N}} \in \mathcal{X}^{\mathcal{N}}} \frac{\langle r(u_N(\mu);\mu), v^{\mathcal{N}} \rangle}{\|u_N(\mu)\|_{\mathcal{X}}}. \tag{12}$$

This is nothing else than computing the Riesz representations in (11) on the truth space $\mathcal{X}^{\mathcal{N}}$, i.e., determine an approximation $\hat{r}_{q,i}^{\mathcal{N}} \in \mathcal{X}^{\mathcal{N}}$ of $\hat{r}_{q,i}$ as

$$\big(\hat{r}_{q,i}^{\mathcal{N}}, v^{\mathcal{N}}\big)_{\mathcal{X}} = \langle r_q(\xi_i), v^{\mathcal{N}} \rangle \qquad \forall v^{\mathcal{N}} \in \mathcal{X}^{\mathcal{N}}. \tag{13}$$

Having these, we compute the corresponding approximation $\boldsymbol{R}_{(q,i),(q',i')}^{\mathcal{N}}$ of $\boldsymbol{R}_{(q,i),(q',i')}$ in (10) and insert this into (9) yielding the approximation $R_N^{\mathcal{N}}(\mu)$ of $R_N(\mu)$. This has an obvious consequence, namely that this does not yield a bound for the exact error as in (5) but only for the *truth error*, i.e.,

$$\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_{\mathcal{X}} \leq \frac{1}{\alpha(\mu)} R_N^{\mathcal{N}}(\mu) =: \Delta_N^{\mathcal{N}}(\mu).$$

2.4 Estimating the Exact Error

From the above derivation, it should be clear that using a common uniform truth will not yield a control of the exact error. In [7], it was suggested to solve (11) adaptively in order reach any desired accuracy. It turned out, however, that due to the sum in (9), an adaptive error control in (11) is not sufficient for an (online-)efficiently computable error estimate. There are alternative approaches using e.g. an adaptive basis generation, [9,10]. To the best of our knowledge, however, these approaches are limited to specific problem classes. In this paper, we present an alternative approach using (multi-)wavelets.

3 Wavelet-Based Error Estimation

We start by reviewing the essentials of wavelet bases that are relevant for the problem at hand. For details we refer to [8] and references therein.

3.1 Wavelet Bases

For simplicity, we restrict ourselves to the univariate case and define the scaled and shifted version of a continuous function $f \in C(\mathbb{R})$ of compact support as (j is a *scaling* or *level*, k is a *shift* or *location* in space)

$$f_{j,k}(x) := 2^{j/2} f(2^j x - k), \qquad x \in \mathbb{R}, \quad j,k \in \mathbb{Z}.$$

A system $\Psi := \{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is called *wavelet system* with d *vanishing moments* and *regularity* $s \in \mathbb{R}_+$ if (1) Ψ is a Riesz basis for $L_2(\mathbb{R})$; (2) $\int_{\mathbb{R}} x^p \, \psi_{j,k}(x)\, dx = 0$ for all $j,k \in \mathbb{Z}$ and $0 \le p \le d-1$; (3) $\psi \in H^s(\mathbb{R})$; (4) $|\operatorname{supp} \psi_{j,k}| \sim 2^j$. The function ψ is called *mother wavelet*. It is a remarkable fact, [2,8], that Ψ is not only a Riesz basis for $L_2(\mathbb{R})$ but allows for a characterization of Sobolev spaces, i.e.,

$$\Big\| \sum_{j,k\in\mathbb{Z}} d_{j,k}\, , \psi_{j,k} \Big\|^2_{H^\sigma(\mathbb{R})} \sim \sum_{j,k\in\mathbb{Z}} 2^{2\sigma j} |d_{j,k}|^2, \qquad \sigma \in (0, \min\{s,d\}).^3$$

Moreover, the Riesz representation theorem ensures the existence of a *dual* wavelet basis $\tilde\Psi := \{\tilde\psi_{j,k} : j,k \in \mathbb{Z}\}$, which is also a wavelet system (for certain parameters $\tilde d$, $\tilde s$) and

$$(\psi_{j,k}, \tilde\psi_{j',k'})_{L_2(\mathbb{R})} = \delta_{j,j'}\, \delta_{k,k'}, \qquad j,j',k,k' \in \mathbb{Z}.$$

The collection $(\Psi, \tilde\Psi)$ of both wavelet systems is called a *biorthogonal wavelet system*. Examples include biorthogonal B-spline wavelets and orthonormal multi-wavelets, which we use here. The dual wavelet system also allows a characterization of Sobolev spaces $H^\sigma(\mathbb{R})$, $-\min\{\tilde s, \tilde d\} < \sigma \le 0$. Note, that these are Sobolev spaces of *negative* order, i.e., *dual* spaces (i.e., those spaces the residual resides in).

There are generalizations for wavelet systems for $L_2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain. To fix notation, we collect shift and scaling index $\lambda = (j,k)$, denote the level by $|\lambda| := j$ and the index range by $\mathcal{J}$. Then, the relevant dual system takes the form $\tilde\Psi := \{\tilde\psi : \lambda \in \mathcal{J}\}$ and the norm equivalence reads for $-s < \sigma \le 0$

$$\|\boldsymbol{d}^T \tilde\Psi\|^2_{H^\sigma(\Omega)} = \Big\| \sum_{\lambda\in\mathcal{J}} d_\lambda \tilde\psi_\lambda \Big\|^2_{H^\sigma(\Omega)} \sim \sum_{\lambda\in\mathcal{J}} 2^{2|\lambda|\sigma} |d_\lambda|^2 =: \|\boldsymbol{D}^\sigma \boldsymbol{d}\|^2_{\ell_2(\mathcal{J})}, \quad (14)$$

where $\boldsymbol{d} := (d_\lambda)_{\lambda\in\mathcal{J}}$ and $\boldsymbol{D} := \operatorname{diag}(2^{2|\lambda|})$.

[3] Here $A \sim B$ abbreviates the existence of constants $0 < c \le C < \infty$ such that $cA \le B \le CB$.

3.2 Wavelet-Based Residual Expansion

The new item presented in this paper is to use (14) for (9). The point of departure is (7). We expand $r_q(w) \in \mathcal{X}'$ by $\tilde{\Psi}$, i.e., $r_q(w) = \sum_{\lambda\in\mathcal{J}} \langle r_q(w), \psi_\lambda\rangle\, \tilde{\psi}_\lambda$, $w \in \mathcal{X}$, and for ξ_i, $1 \le i \le N$, we set $d_{(q,i),\lambda} := \langle r_q(\xi_i), \psi_\lambda\rangle$. Then, by (7) and (8)

$$\begin{aligned} r(u_N(\mu);\mu) &= \sum_{q=1}^{Q_r} \vartheta_q^r(\mu)\, r_q(u_N(\mu)) = \sum_{q=1}^{Q_r}\sum_{i=1}^{N}\sum_{\lambda\in\mathcal{J}} \vartheta_q^r(\mu)\, c_i(\mu)\, d_{(q,i),\lambda}\, \tilde{\psi}_\lambda \\ &=: \sum_{\lambda\in\mathcal{J}} \Big(\sum_{m\in M} \sigma_m(\mu)\, d_{m,\lambda} \Big) \tilde{\psi}_\lambda =: \sum_{\lambda\in\mathcal{J}} r_\lambda(\mu)\, \tilde{\psi}_\lambda, \end{aligned}$$

where $M := \{1,\dots,Q_r\} \times \{1,\dots,N\}$, $m := (q,i)$ and $\sigma_m(\mu) := \vartheta_q^r(\mu)\, c_i(\mu)$. In order to estimate $R_N(\mu) := \|r(u_N(\mu);\mu)\|_{\mathcal{X}'}$, we have

$$\begin{aligned} R_N(\mu)^2 &\sim \sum_{\lambda\in\mathcal{J}} 2^{-2|\lambda|}\, |r_\lambda(\mu)|^2 = \sum_{\lambda\in\mathcal{J}} 2^{-2|\lambda|} \Big| \sum_{m\in M} \sigma_m(\mu)\, d_{m,\lambda} \Big|^2 \\ &= \sum_{\lambda\in\mathcal{J}} 2^{-2|\lambda|} \sum_{m,m'\in M} \sigma_m(\mu)\, \sigma_{m'}(\mu)\, d_{m,\lambda}\, d_{m',\lambda} \\ &= \sum_{m,m'\in M} \sigma_m(\mu)\, \sigma_{m'}(\mu) \sum_{\lambda\in\mathcal{J}} 2^{-2|\lambda|} d_{m,\lambda}\, d_{m',\lambda} \\ &=: \sum_{m,m'\in M} \sigma_m(\mu)\, \sigma_{m'}(\mu) s_{m,m'} =: R_N^{\Psi}(\mu)^2, \end{aligned} \tag{15}$$

where we abbreviate $s_{m,m'} := \sum_{\lambda\in\mathcal{J}} 2^{-2|\lambda|} d_{m,\lambda}\, d_{m',\lambda}$. These terms are parameter-independent and can be precomputed offline at any desired accuracy, which can be seen as follows. Since $r_q(\xi) \in \mathcal{X}'$, we have $(2^{-|\lambda|} d_{m,\lambda})_{\lambda\in\mathcal{J}} \in \ell_2(\mathcal{J})$, which means that there is a decay with respect to the level. In fact, the following Whitney-type estimate is well-known

$$|d_{m,\lambda}| = |\langle r_q(\xi_i), \psi_\lambda\rangle| \le C\, 2^{-\sigma|\lambda|}\, \|r_q(\xi_i)\|_{H^\sigma(\operatorname{supp}\psi_\lambda)}, \quad 0 \le \sigma < d, \tag{16}$$

with d as described in §3.1. This means that we have an *a priori* estimate for the size of each $d_{m,\lambda}$ in terms of the (known) local Sobolev regularity, which in turn implies that the sum over infinitely many terms in $s_{m,m'}$ can be truncated a priori to a finite one with an a priori bound. This shows that we can indeed precompute $s_{m,m'}$ at any desired tolerance.

Finally, the corresponding error estimator is $\Delta_N^{\Psi}(\mu) := (c_\Psi\, \alpha(\mu))^{-1} R_N^{\Psi}(\mu)$, where c_Ψ is the lower equivalence constant in (14).

4 Numerical Experiments

In this section, we compare the numerical performance of the wavelet-based error estimator $\Delta_N^{\Psi}(\mu)$ in (15) with the standard RB-truth-based $\Delta_N^{\mathcal{N}}(\mu)$ in

(12) (implemented in RB-Matlab, [3]) and the exact error. For this purpose, we consider the simple Laplace problem on $\Omega = (0,1)^2$ with a parameter-dependent source $f(\mu) \in H^{-1}$, i.e., for $x = (x_1, x_2) \in \Omega$, $\mu = (\mu_1, \mu_2) \in \mathcal{D} := [0.2, 0.8]^2$, $a := 1/35$, we set

$$f(x;\mu) := -e^{-\frac{(x_1-\mu_1)^2+(x_2-\mu_2)^2}{a^2}} \left(\frac{4}{a^4}(x_1-\mu_1)^2 + \frac{2}{a^2}(x_2-\mu_2)^2 - \frac{4}{a^4} \right),$$

and let $u(\mu)$ be the corresponding exact solution, which can be computed analytically. Even though $u(\mu) \in C^\infty(\Omega)$, it has a steep gradient (at μ-dependent locations) such that many basis functions are necessary to resolve local details sufficiently well for all $\mu \in \mathcal{D}$. We compare 3 scenarios, namely (1) a truth discretization consisting of 37,249 cubic finite elements (realizing a snapshot tolerance of $\varepsilon = 10^{-2}$); (2) the same with 2,362,369 elements (with $\varepsilon = 10^{-4}$) and (3) an adaptive cubic multi-wavelet snapshot generation as in [1] (where we set the tolerance to 10^{-5}, which leads to 80,637 wavelets at most). In all cases, the RB space X_N of dimension $N = 6$ is computed by a weak greedy on $\mathcal{D}^{\text{train}} := \{0.2, 0.5, 0.8\}^2$ with snapshot orthonormalization.

The results are summarized in Table 1. We report average errors over the test set $\mathcal{D}^{\text{test}} := \{0.2, 0.3, \ldots, 0.8\}^2 \subset \mathcal{D}$. Of course, in case of a truth discretization, the *truth error* $\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_{\mathcal{X}}$ and hence the standard RB error estimator $\Delta_N^{\mathcal{N}}(\mu)$ is zero (or machine accuracy) for snapshot parameters $\mu \in S_N$. The *exact error* $\|u(\mu) - u_N(\mu)\|_{\mathcal{X}}$ does not vanish but is in the order of the approximation tolerance of the truth space $\mathcal{X}^{\mathcal{N}}$. For the non-snapshot parameters we observe an effectivity of about 1.94 – as compared to the truth error. The exact error is in the same range and shows that even a highly resolved FE mesh is not capable to guarantee a sufficient accuracy for all parameters.

$\varnothing$ error	case	exact $\|u(\mu) - u_N(\mu)\|_{\mathcal{X}}$	truth $\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_{\mathcal{X}}$	$\Delta_N^{\Psi}(\mu)$	$\Delta_N^{\mathcal{N}}(\mu)$
S_N	1	$9.6 \cdot 10^{-3}$	$2.7 \cdot 10^{-15}$	—	0
	2	$9.2 \cdot 10^{-5}$	$1.6 \cdot 10^{-13}$	—	0
	3	$2.2 \cdot 10^{-6}$	—	$3.6 \cdot 10^{-6}$	—
$\mathcal{D} \setminus S_N$	1	$1.8 \cdot 10^{0}$	$1.8 \cdot 10^{0}$	—	$3.5 \cdot 10^{0}$
	2	$1.8 \cdot 10^{0}$	$1.8 \cdot 10^{0}$	—	$3.5 \cdot 10^{0}$
	3	$1.7 \cdot 10^{0}$	—	$3.1 \cdot 10^{0}$	—

Table 1. Average error for snapshot (S_N) and non-snapshot ($\mathcal{D} \setminus S_N$) parameters over $\mathcal{D}^{\text{test}}$. Case 1: $\mathcal{X}^{\mathcal{N}}$ is cubic FEM grid with 37,249 dof; Case 2: $\mathcal{X}^{\mathcal{N}}$ is cubic FEM grid with 2,362,369 dof; Case 3: Adaptive snapshots based upon cubic multi-wavelets with at most 80,637 dof.

In case (3), using adaptive snapshot computation, the tolerance is chosen a-priori, which is achieved both by the exact error and the error estimator. This is a bound for the exact error rather than for the truth error (and with much fewer d.o.f.). We obtain an effectivity of about 1.8, i.e., in the same order as in the truth case, but now for the exact error. The online complexity for the wavelet-based estimator $\Delta_N^{\Psi}(\mu)$ is $\mathcal{O}(N^2 + N^2 Q_a^2 + N Q_f + Q_f^2)$ (with Q_a, Q_f as in (6) and $Q_a = 1$, $Q_f = 49$ in this case, $N = 6$), which is online efficient. The CPU time for the wavelet-based error estimator is even too small to measure it with standard tools[3].

We conclude that the wavelet-based error estimator is both theoretically and practically efficient, reliable and effective. Moreover, it allows a control of the exact error.

References

1. M. Ali, K. Steih, and K. Urban, *Reduced basis methods based upon adaptive snapshot computations*, 2014, arXiv:1407.1708.
2. W. Dahmen, *Wavelet and multiscale methods for operator equations*, Acta Numerica, 1997, Acta Numer., vol. 6, Cambridge Univ. Press, Cambridge, 1997, pp. 55–228.
3. M. Drohmann, B. Haasdonk, and M. Ohlberger, *Rbmatlab*, 2013, `http://www.ians.uni-stuttgart.de/MoRePaS/software/rbmatlab`.
4. J. Hesthaven, G. Rozza, and B. Stamm, *Certified reduced basis methods for parametrized partial differential equations*, SpringerBriefs in Mathematics, vol. Mathematics, Springer, Cham, 2016.
5. D. Huynh, G. Rozza, S. Sen, and A. Patera, *A successive constraint linear optimization method for lower bounds of parametric coercivity and inf-sup stability constants*, C. R. Math. Acad. Sci. Paris **345**:8 (2007), 473–478.
6. A. Quarteroni, A. Manzoni, and F. Negri, *Reduced basis methods for partial differential equations*, Unitext, vol. 92, Springer, Cham, 2016.
7. K. Steih, *Reduced basis methods for time-periodic parametric partial differential equations*, Ph.D. thesis, Ulm University, 2014.
8. K. Urban, *Wavelet methods for elliptic partial differential equations*, Oxford University Press, 2009.
9. M. Yano, *A Reduced Basis Method with Exact-Solution Certificates for Symmetric Coercive Equations*, Comp. Meth. Appl. Mech. Engin. **287** (2015), 290–309.
10. ———, *A minimum-residual mixed reduced basis method: Exact residual certification and simultaneous finite-element reduced-basis refinement*, Math. Model. Numer. Anal. **50**:1 (2015), 163–185.

[3] The computation is finishied before C++'s `ctime std::clock function` manages to update the number of clocks.