Space-Time Reduced Basis Methods for Time-Periodic Partial Differential Equations

Kristina Steih, Ulm University, Germany *
Karsten Urban, Ulm University, Germany *

* *Institute for Numerical Mathematics, Ulm University, Germany, (email: {kristina.steih, karsten.urban}@uni-ulm.de).*

Abstract: We consider parameter-dependent time-periodic parabolic problems as they occur e.g. in the modeling of rotating propellers. The goal is to determine the parameters (e.g. design or steering parameters) such that a given output of interest (e.g. the efficiency of the propeller) is maximized. A standard approach to numerically solve time-periodic problems is a time-stepping (fixed-point) scheme. This approach, however, often suffers from some drawbacks, in particular within the Reduced Basis Method (RBM). First, there might be long transient phases before a periodic or steady state is reached, which is particularly disadvantageous in the online phase. Moreover, corresponding error estimates usually include sums over time-steps which might become inaccurate.
Instead, we consider a space-time variational formulation using periodic basis functions in time, which avoids the need for fixed-point iterations. Based on this variational formulation, we have developed a space-time RBM using wavelets in time and derived corresponding a-posteriori error estimates. We present numerical results indicating the efficiency of the method as well as the effectivity of the derived error bounds.

Keywords: Reduced-order models, error estimation, time-periodicity, space-time.

1. INTRODUCTION

Let H be a Hilbert space, $V \hookrightarrow H \hookrightarrow V'$ a Gelfand triple and $\mathcal{D} \subset \mathbb{R}^p$ a parameter set. We are interested in outputs $J(\mu) := \int_0^T \ell(u(t;\mu))dt$, $\mu \in \mathcal{D}$, with a linear functional $\ell : V \to \mathbb{R}$ and $u(\cdot;\mu)$ being the solution over the time interval $I := (0,T)$ of the time-periodic parametrized partial differential equation (PPDE)

$$u_t(t;\mu) + A(t;\mu)u(t) = g(t;\mu) \quad \text{in } V',\ t \in (0,T), \qquad (1)$$
$$u(0;\mu) = u(T;\mu) \quad \text{in } H. \qquad (2)$$

Here, $g(\cdot;\mu) \in L_2(I;V')$ is given and $A(t;\mu) \in \mathcal{L}(V,V')$ is defined by $\langle A(t;\mu)u, v\rangle := a(t;u,v;\mu)$ for $v \in V$, $\mu \in \mathcal{D}$, $t \in I$, the duality pairing $\langle\cdot,\cdot\rangle$ in $V' \times V$ and a continuous and coercive bilinear form $a(t;\cdot,\cdot,\mu) : V \times V \to \mathbb{R}$ with coercivity and continuity constants $\alpha(\mu) \geq \alpha_0$, $\gamma(\mu) \leq \gamma_0$ uniformly in $\mathcal{D}$ and I.

Such problems are relevant e.g. for all kinds of rotators and propellers, with the parameter $\mu \in \mathcal{D}$ modeling design or steering properties and $J(\mu)$ representing the efficiency or some other time-averaged physical quantity.

A standard approach to numerically solving (1),(2) is by a fixed-point iteration, i.e. replacing (2) by an initial condition and solving a sequence of such initial value problems by using the approximation at the final time T as initial value for the next iteration. Depending on the speed of convergence of such a fixed-point iteration, this may require several numerical solutions of the initial value problem corresponding to (1).

Moreover, we are often interested in the optimization of the output w.r.t the parameters $\mu \in \mathcal{D}$. Hence, e.g. in a numerical optimization, the PPDE has to be solved for several values of the parameter. This is a typical *multi-query* situation where Reduced Basis Methods (RBMs) can be applied to construct a reduced model that can be solved highly efficiently and gives rise to a-posteriori error control. In such a framework, the fixed-point iteration would have to be performed also online with possibly large additional costs.

There are several papers considering RBMs for evolution equations, e.g. Dihlmann et al. (2011); Grepl (2012); Grepl and Patera (2005); Haasdonk and Ohlberger (2008); Rovas et al. (2006). Within the fixed-point framework, we have introduced error bounds and efficient (offline) routines for the RBM in Steih (2010). This approach, however, suffers from two drawbacks, namely the possibly large number of fixed-point iterations and the possible increase of the error within the iteration. The reason is that conventional time-stepping RBMs for time-dependent problems lead to time-*independent* bases (cf. Grepl and Patera (2005); Haasdonk and Ohlberger (2008)). Within those frameworks, error bounds can only be formulated for discrete spatio-temporal norms and involve sums over all time steps, thus growing in time. Moreover, in case of time-variant operators, the construction of the RB basis re-

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quires either storage of additional information at each time point (Dihlmann et al. (2011)) or additional computational effort to separate time and space (Grepl (2012)). Instead, we propose a space-time variational formulation combined with the RBM.

2. SPACE-TIME FORMULATION

With $\mathcal{Y} := L_2(I;V)$ and $\mathcal{X} := L_2(I;V) \cap H^1_{\text{per}}(I;V')$ where $H^1_{\text{per}}(I;V') := \{v \in \mathcal{Y} : v_t(t) \in V' \; \forall\, t \in I, v(0) = v(T) \text{ in } H\}$, a variational formulation in space-time of (1) reads:

$$\text{Find } \; u(\mu) \in \mathcal{X} : \; b(u,v;\mu) = f(v;\mu) \quad \forall\, v \in \mathcal{Y}, \tag{3}$$

where

$$b(u,v;\mu) := \int_0^T \langle u_t(t;\mu), v(t)\rangle dt + \int_0^T a(t; u(t,\mu), v(t);\mu) dt,$$
$$f(v;\mu) := \int_0^T (g(t;\mu), v(t))_H dt.$$

Lemma 1. The bilinear form b in (3) is bounded, surjective and satisfies an inf-sup-condition, so that (3) is well-posed.

Proof. Boundedness follows from the continuity of a:

$$|b(u,v)| \le \sqrt{2} \max\{1, \gamma(\mu)\} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}.$$

The special choice of $v_w(\mu) := A^{-1}_{\text{adj}}(\mu) w_t \in \mathcal{Y}$ with the adjoint operator $A_{\text{adj}}(\mu)$ of $A(\mu)$ for each $w \in \mathcal{X}$ yields the inf-sup condition with lower bound

$$\beta_{\text{LB}}(\mu) = \frac{\alpha(\mu) \min\{1, \gamma^{-2}(\mu)\}}{\sqrt{2}\max\{1, \alpha^{-1}(\mu)\}}. \tag{4}$$

Finally, surjectivity follows from the existence of a $z \in \mathcal{X}$ with

$$b(z,v) = \int_0^T a(t; v, v; \mu) dt \ge \alpha(\mu)\|v\|_{\mathcal{Y}} > 0 \qquad \forall\, v \in \mathcal{Y},$$

which can be proven using a sequence of Galerkin approximations together with different a-priori estimates and corresponding convergence results For details, we refer to Urban and Patera (2011), Steih and Urban (2011). □

3. REDUCED BASIS METHOD

The general RBM framework can be summarized as follows: Based on a (suitably fine and thus usually high-dimensional) discretization $\mathcal{X}^{\mathcal{N}}$ of $\mathcal{X}$ (the so-called *truth approximation*), RBMs use so-called snapshots $\{u(\mu_i), i = 1, \dots, N\}$ of the solution to construct a basis for the low-dimensional approximation space $\mathcal{X}_N$, $N \ll \mathcal{N}$, in an *offline phase*. Evaluations of both solution and error estimates for new parameters in the *online phase* then only depend on N, as one considers the Galerkin approximation

$$u_N(\mu) \in \mathcal{X}_N : \; b(u_N, v_N; \mu) = f(v_N;\mu) \quad \forall\, v_N \in \mathcal{Y}_N. \tag{5}$$

The RB framework requires the existence of an affine decomposition of the involved operators, i.e. $b(u,v;\mu) = \sum_{q=1}^{Q_b} \theta_b^q(\mu) b^q(u,v)$, $f(v;\mu) = \sum_{q=1}^{Q_f} \theta_f^q(\mu) f^q(v)$. This decomposition allows the precomputation of parameter-independent quantities in the offline phase and reduces the effort for assembling the N-dimensional systems in the online phase to rapid evaluations of the functions $\theta_\circ^q(\mu)$ and simple matrix-vector multiplications. In case of non-affine dependencies, operators can be approximated by empirical interpolation methods (EIM, Barrault et al. (2004)).

Note that RB methods within time-stepping frameworks additionally require separability of time and space, i.e. affine decompositions of the form $a(t,u,v;\mu) = \sum_{q=1}^{Q_a} \theta_a^q(t,\mu) a^q(u,v)$. This should be kept in mind also for our numerical results reported below.

3.1 Basis generation

In the space-time framework, the snapshots for building the reduced basis are solutions of (3) and thus space-time functions. During the so-called *training* phase, the corresponding parameter values μ_i are selected by a Greedy scheme over a chosen training set Ξ_{train}, based upon a-posteriori error estimators that are rigorous bounds for the error $e_N(\mu) := u(\mu) - u_N(\mu)$. The set of snapshots, orthogonalized with respect to the inner product on $\mathcal{X}$, forms the basis for the approximation space $\mathcal{X}_N$. An online approximation $u_N(\mu) = u_N(\mu)(t,x)$ for a new parameter is then obtained as the solution of the $N \times N$ system corresponding to (5). The computational effort is $\mathcal{O}(N^3)$ as the RB system matrices are usually densely populated.

Fixed-point approaches are an extension of well-known time-stepping methods to obtain periodic or stationary solutions. For ease of presentation, we consider the backward Euler method. A truth solution then consists of finding $u^k(\mu) = u(t_k,\mu) \in V$ for the discrete time points $\{t_i := i\Delta t\}_{i=0,\dots,K}$ with $u^0(\mu) = u^K(\mu)$ and

$$(u^k(\mu), w)_H + \Delta t\, a(t_k, u^k(\mu), w; \mu)$$
$$= (u^{k-1}(\mu), w)_H + \Delta t\, g(t_k, w; \mu) \quad \forall\, w \in V,$$

for $k = 1, \dots, K$. Starting with an (arbitrary) $u^0_{(0)}(\mu)$, one solves the above initial value problem (IVP) and uses $u^0_{(1)}(\mu) := u^K_{(0)}(\mu)$ for the next iteration, stopping when $\|u^0_{(M)}(\mu) - u^0_{(M-1)}(\mu)\|_V \le tol$. In the following, we denote by $M = M(tol)$ the required number of *fixed-point iterations* until convergence.

Basis generation is usually done with a POD-Greedy approach: After the N-th snapshot $\{u^k(\mu), k = 0, \dots, K\}$ is chosen greedily using an error estimator, the error of the projection onto the existing $(N-1)$-dimensional basis is subjected to a POD and the first mode taken as N-th basis function of the reduced basis V_N so that this is a basis in space only.

The corresponding RB problems are to find solutions $\{u_N^k(\mu), i = 0, \dots, K\}$ with $u_N^i \in V_N$, $u_N^0(\mu) = u_N^K(\mu)$ and

$$(u_N^k(\mu), w_N)_H + \Delta t\, a(t_k, u_N^k(\mu), w_N; \mu)$$
$$= (u_N^{k-1}(\mu), w_N)_H + \Delta t\, g(t_k, w_N; \mu) \quad \forall\, w_N \in V_N,$$

for $k = 1, \dots, K$. Hence, in the online phase another fixed-point problem has to be solved and the computational effort for a reduced problem with this approach is $\mathcal{O}(MKN^3)$.

3.2 A-posteriori errors

All RB basis generation methods rely on a-posteriori estimators that are not only required to be rigorous upper bounds for the error, but must also be rapidly evaluable, i.e., their computation has to be $\mathcal{N}$-independent. This property enables the otherwise computationally exhaustive Greedy to efficiently search over the training set Ξ_{train}.

A modification of the known error bound for parabolic problems (Grepl and Patera (2005)) yields the following result for periodic solutions using the fixed-point method.

Lemma 2. The error $e_N(\mu)$ in the fixed-point approach can be bounded by

$$\left(\Delta t\sum_{k=1}^{K}\|e_N^k(\mu)\|_V^2\right)^{\frac{1}{2}} \leq \left(\frac{\Delta t}{\alpha^2(\mu)}\sum_{k=1}^{K}\|r_N^k(\cdot;\mu)\|_{V'}^2\right)^{\frac{1}{2}} \quad (6)$$

with $r_N^k(w;\mu) := \Delta t\, g(t_k,w;\mu) - (u_N^k(\mu)-u_N^{k-1}(\mu),w)_H - \Delta t\, a(t_k,u_N^k(\mu),w;\mu)$ the residual at time step k.

Note that the left hand side $(\Delta t\sum_{k=1}^{K}\|e_N^k(\mu)\|_V^2)^{1/2}$ is a trapezoidal approximation of the norm $\|e_N(\mu)\|_{\mathcal{Y}}$, as due to periodicity we have $e_N^0(\mu) = e_N^K(\mu)$.

The space-time error bounds by their very nature do not involve time-steps and are thus much more similar to the well-known estimates in the elliptic case.

Lemma 3. The error of the space-time RB approximation $u_N(\mu)$ in $\mathcal{Y}$ can be bounded as follows:

$$\|e_N(\mu)\|_{\mathcal{Y}} \;\leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\alpha(\mu)},$$

where $r_N(v;\mu) := f(v;\mu) - b(u_N(\mu),v;\mu)$.

Proof. Let $u \in \mathcal{X}$. Periodicity of u and coercivity of a yield

$$b(u,u;\mu) = \frac{1}{2}\left(\|u(T)\|_H^2 - \|u(0)\|_0^2\right) + \int_0^T a(t,u,u;\mu)dt$$
$$\geq \int_0^T \alpha(\mu)\|u\|_V^2 dt \;=\; \alpha(\mu)\,\|u\|_{\mathcal{Y}}^2. \quad (7)$$

As $u^{\mathcal{N}} \in \mathcal{X}$, $u_N \in \mathcal{X}^N \subset \mathcal{X}$, we know that $e_N(\mu) \in \mathcal{X}$. Using (7) and $b(e_N(\mu),v;\mu) = r_N(v;\mu)$, we then have

$$\alpha(\mu)\|e_N(\mu)\|_{\mathcal{Y}}^2 \leq b(e_N(\mu),e_N(\mu);\mu) = r_N(e_N(\mu);\mu)$$
$$\leq \|r_N(\cdot;\mu)\|_{\mathcal{Y}'}\|e_N(\mu)\|_{\mathcal{Y}},$$

which yields the claim. □

Moreover, the space-time framework additionally allows the derivation of an error bound in the solution space $\mathcal{X}$, which is not possible in time-stepping contexts.

Lemma 4. The error of the space-time RB approximation $u_N(\mu)$ in $\mathcal{X}$ can be bounded by

$$\|e_N(\mu)\|_{\mathcal{X}} \;\leq \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\beta(\mu)}, \quad (8)$$

where $\beta(\mu)$ is the inf-sup constant of the bilinear form $b(\cdot,\cdot;\mu)$.

Proof. From Lemma 1, we know that the inf-sup constant $\beta(\mu)$ exists. Its definition yields

$$\beta(\mu) = \inf_{0\neq u\in\mathcal{X}}\sup_{0\neq v\in\mathcal{Y}}\frac{|b(u,v;\mu)|}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} = \inf_{0\neq u\in\mathcal{X}}\frac{\|b(u,\cdot;\mu)\|_{\mathcal{Y}'}}{\|u\|_{\mathcal{X}}},$$

so that

$$\beta(\mu) \leq \frac{\|b(e(\mu),\cdot;\mu)\|_{\mathcal{Y}'}}{\|e_N(\mu)\|_{\mathcal{X}}} = \frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\|e_N(\mu)\|_{\mathcal{X}}}. \qquad \square$$

The space-time bound of the output approximation error involves the dual problem

$$\text{Find}\quad z(\mu)\in\mathcal{Y}:\; b(u,z;\mu) = J(u) \quad \forall\, u\in\mathcal{X}$$

with corresponding approximation in the dual reduced spaces $\mathcal{X}_{N^*}$, $\mathcal{Y}_{N^*}$. We then obtain a slightly generalized Petrov-Galerkin version of the output bound from Rovas et al. (2006).

Lemma 5. With the dual reduced solution $z_{N^*}(\mu)$, dual residual $r_{N^*}^*(w;\mu) := J(w) - b(w,z_{N^*}(\mu);\mu)$ and output approximation $J_{N,N^*}(\mu) := J(u_N(\mu)) - r_N(z_{N^*}(\mu);\mu)$, the output error is bounded by

$$|J(u(\mu)) - J_{N,N^*}(\mu)| \leq \frac{1}{\beta(\mu)}\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}\|r_{N*}^*(\cdot;\mu)\|_{\mathcal{X}'}.$$

Proof. With the linearity of the output and (8), we have

$$|J(u(\mu)) - J_{N,N^*}(\mu))| = |J(e_N(\mu);\mu) - r_N(z_{N^*}(\mu);\mu)|$$
$$= |J(e_N(\mu)) - b(e_N(\mu),z_{N^*}(\mu);\mu)|$$
$$= |r_{N^*}^*(e_N(\mu);\mu)|$$
$$\leq \|r_{N^*}^*(\cdot;\mu)\|_{\mathcal{X}'}\|e_N(\mu)\|_{\mathcal{X}},$$
$$\leq \|r_{N^*}^*(\cdot;\mu)\|_{\mathcal{X}'}\frac{\|r_N(\cdot;\mu)\|_{\mathcal{Y}'}}{\beta(\mu)}. \qquad \square$$

As typical in RB methods, these error bounds require the computation of the dual norms of the involved residuals. This is usually done by calculating the respective Riesz representations, i.e. the solutions $\hat{\varepsilon}_N(\mu)$ of e.g. $r_N(v;\mu) = (\hat{\varepsilon}_N(\mu),v)_{\mathcal{Y}}$, and exploiting that $\|r_N(\cdot;\mu)\|_{\mathcal{Y}'} = \|\hat{\varepsilon}_N(\mu)\|_{\mathcal{Y}}$. An offline-online decomposition can be utilized for the computation of $\|\hat{\varepsilon}_N(\mu)\|_{\mathcal{Y}}$, where the offline stage requires the solution of $Q_f + NQ_a$ space-time problems. The same number of eigenproblems in V is needed for the fixed-point bound (6), but only if a and g are either time-independent or separable into time and space components. Otherwise, individual Riesz representations have to be calculated for each time step $i = 1,\ldots,K$.

Another important aspect of an error bound $\Delta_N(\mu)$ is its effectivity

$$\eta(\mu) := \frac{\Delta_N(\mu)}{\|e_N(\mu)\|}.$$

The rigor of Δ_N ensures that $\eta(\mu) \geq 1$. On the other hand, to avoid an overestimation of the true error - which leads to large reduced bases and hence an unnecessary online computational effort, and might even hinder the Greedy search for optimal snapshot parameters - the effectivity should be as close to unity as possible.

Finally, computing $\alpha(\mu)$ and $\beta(\mu)$ requires the solution of generalized space-time eigenproblems, as

$$\alpha(\mu) = \inf_{w\in\mathcal{X}}\frac{b(w,w;\mu)}{\|w\|_{\mathcal{Y}}^2}, \quad \beta(\mu) = \inf_{w\in\mathcal{X}}\frac{\|T(\mu)w\|_{\mathcal{Y}}}{\|w\|_{\mathcal{X}}},$$

with the supremizing operator $T(\mu)w = \operatorname{argsup}_{v\in\mathcal{Y}}\frac{b(w,v;\mu)}{\|v\|_{\mathcal{Y}}}$. An offline-online decomposition for the computation of lower bounds $\alpha_{\text{LB}}(\mu) \leq \alpha(\mu)$, $\beta_{\text{LB}}(\mu) \leq \beta(\mu)$ can be achieved with the Successive Constraint Method (SCM, Huynh et al. (2007)).

As in (4), sometimes also analytical bounds can be derived. Note that for time-dependent linear forms in a fixed-point framework we have to work with the time-dependent constants $\alpha(t^k,\mu)$ for $k = 1,\ldots,K$. Any search for a lower bound, e.g. $\alpha_{LB}(\mu) \leq \alpha(t^k,\mu)$, can then only be done with respect to these discrete times.

So far, RBM for space-time approaches has only be considered in Rovas et al. (2006), where a-posteriori estimates for the output have been derived. There, the authors consider a pure Galerkin formulation in a less general setting and employ Discontinuous Galerkin methods in order to cope with the size of full space-time discretizations.

4. WAVELET METHODS

The simultaneous treatment of space and time effectively increases the problem dimension by one. Hence, in order to compute truth solutions in the space-time approach, a good and efficient discretization is paramount. Both $\mathcal{X} = (L_2(I) \otimes V) \cap (H^1_{\text{per}}(I) \otimes V')$ and $\mathcal{Y} = L_2(I) \otimes V$ have tensor product representations which can be mirrored in the structure of the discrete approximation spaces, i.e. $\mathcal{X}^{\mathcal{N}} = S_h \otimes V_h$, $\mathcal{Y}^{\mathcal{N}} = T_h \otimes V_h$ for some spaces $S_h \subset H^1_{\text{per}}(I)$, $T_h \subset L_2(I)$, $V_h \subset V$.

Wavelet discretizations have the advantage that periodic basis functions are easily constructed. Moreover, certain norm equivalence results enable the simple evaluation of norms in Sobolev spaces $H^s(\Omega)$ for all s in a given range that depends on the wavelet construction. Let $\Theta := \{\vartheta_i, i \in \mathcal{I}\}$ be a collection of wavelets that form a (normalized) Riesz basis of $L_2(I)$ and after renormalization one of $H^1(I)$. Similarly, let $\Sigma := \{\sigma_j, j \in \mathcal{J}\}$ be a normalized Riesz basis of H and renormalizable to Riesz bases in V and V'. Then $\Theta \otimes \Sigma$ can be renormalized in $\mathcal{X}$ as well as $\mathcal{Y}$ to form a basis of both spaces, respectively (see Schwab and Stevenson (2009)):

$$\mathcal{Y}^{\mathcal{N}} = \left\{ (t,x) \mapsto \frac{\vartheta_i(t)\sigma_j(x)}{\|\sigma_j\|_V}, i \in \mathcal{I}, j \in \mathcal{J} \right\},$$

$$\mathcal{X}^{\mathcal{N}} = \left\{ \frac{\vartheta_i(\cdot)\sigma_j(\cdot)}{\sqrt{\|\sigma_j\|_V^2 + \|\vartheta_i\|^2_{H^1(I)}\|\sigma_j\|^2_{V'}}}, i \in \mathcal{I}, j \in \mathcal{J} \right\}.$$

We propose the use of (periodized) linear B-spline wavelets for the time-discretization. Due to the tensor product structure of trial and test spaces with respect to space and time we are left with the freedom of choosing any appropriate spatial discretization that might be suitable for the problem at hand. Of course, one might use any temporal ansatz function, e.g. trigonometric polynomials. We choose wavelets since they allow for adaptive methods with proven optimal convergence rate (Schwab and Stevenson (2009), Cohen et al. (2001)).

5. NUMERICAL RESULTS

In the following, we apply both the space-time approach as well as the fixed-point methods to some numerical examples, using LAWA (2011) for the wavelet-based space-time calculations and the finite element library `libMesh/rbOOmit` (Kirk et al. (2006)) for the fixed-point calculations. Just for simplicity, we consider problems that are 1D in space and use a linear Dijkema wavelet construction for the approximation in space (Dijkema (2009)).

We restrict ourselves to problems with affine structures in the tensor form $\sum_{q=1}^{Q} \theta^q(\mu) b^{q,t}(t) b^{q,x}(x)$, i.e. without actual space-time interdependence, as such problems can be treated with both space-time and fixed-point methods. Note that this is a somewhat unfair situation for the space-time approach, since such a separation allows for completely independent discretizations in space and time. This also concerns our choice of wavelets in time, which has been motivated by the easy realization of periodicity and of adaptive methods. The latter, however, is not beneficial in this particular example. In fact, adaptivity only pays off if the solution allows for a sparse representation so that only few basis functions suffice for a representation up to a given error tolerance. If the problem is affine separable with respect to time, however, adaptivity usually is not needed. In fact, sources for sparse representations are (i) the operator, (ii) the geometry of the domain and (iii) the right-hand side of the problem. In 1D, however, (i) and (ii) are not relevant and singularities caused by the right-hand side can also be resolved without adaptive schemes. But even using non-adaptive discretizations, the tensor product structure of the examples is not favorable for the space-time approach as it does not expose its ability to capture non-separable space-time interaction and dependencies. These facts have to be kept in mind when interpreting our numerical experiments.

As discretization we employed a preconditioned Wavelet-Galerkin scheme with wavelets of level $J = 6$ for the first and $J = 5$ for the second example in both time and space and chose linear finite elements and time steps of corresponding size for the fixed point methods.

5.1 Heat equation

In order to test our approach, we first consider the simple example of a heat equation $u_t - u_{xx} = g(\mu)$ on $[0, T] \times [0, 1]$ with homogeneous Dirichlet boundary conditions and known reference solution $u(t, x) = e^{-60(x-1/2)^2}(x - \mu c(t))^2$, $c(t) = \frac{1}{2} + \frac{1}{4}\sin(2\pi t)$, $\mu \in \mathcal{D} := [0.5, 2]$. In this case, the right hand side has the form $g(t, x; \mu) = \sum_{i=0}^{2} \mu^i g^{(i)}(t, x)$, so that three basis functions are sufficient to span the solution manifold. In Fig. 1, it can be seen that the space-time Greedy training indeed yields this minimum number of basis functions. As all linear and bilinear forms are space-time separable, we can employ a fixed-point approach in comparison for this example by rearranging the affine decomposition to $g(t, x; \mu) = \sum_{i=0}^{4} \theta^{(i)}(t, \mu)\tilde{g}^{(i)}(x)$. However, the POD-Greedy training needs more than the optimal 5 functions to incorporate all temporal information into the reduced basis, cf. Fig. 1. Note that this significantly increases the online complexity.

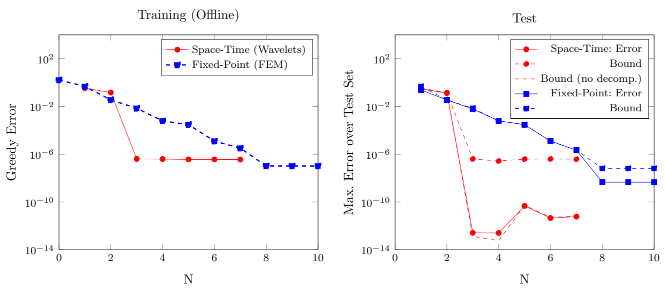


Fig. 1. Comparison of space-time and fixed-point approach for the heat equation example.

Moreover, a computation of the maximal error $\|e_N(\mu)\|_{\mathcal{Y}}$ and the corresponding error bound $\Delta_N^{\text{per}}(\mu)$ for both approaches over a parameter test set $\Xi_{\text{test}} \subset \mathcal{D}$ with $|\Xi_{\text{test}}| =$

49 reveals that the effectivity in both approaches is very good (Fig. 1, right). Note that the error bounds in the offline-online decomposition involve square roots and are thus bounded from below by the root of the machine precision. For the space-time approach, we additionally show the bounds without such a decomposition, i.e. using the Riesz representation of the complete parameter-dependent residual, which can then be computed up to machine precision.
In Fig. 2, the comparison of the maximum effectivities over a test set for different N reveals that the space-time effectivities are in the range $[1.09, 1.12]$, whereas $\max_{\mu\in\mathcal{D}} \eta(\mu) \approx 2$ in the fixed-point calculations. Moreover, the detailed presentations for $N = 1, 2$ show that the effectivities are less parameter-dependent in space-time, which ensures the correct choice of snapshots in the Greedy algorithm and hence a good training behaviour.

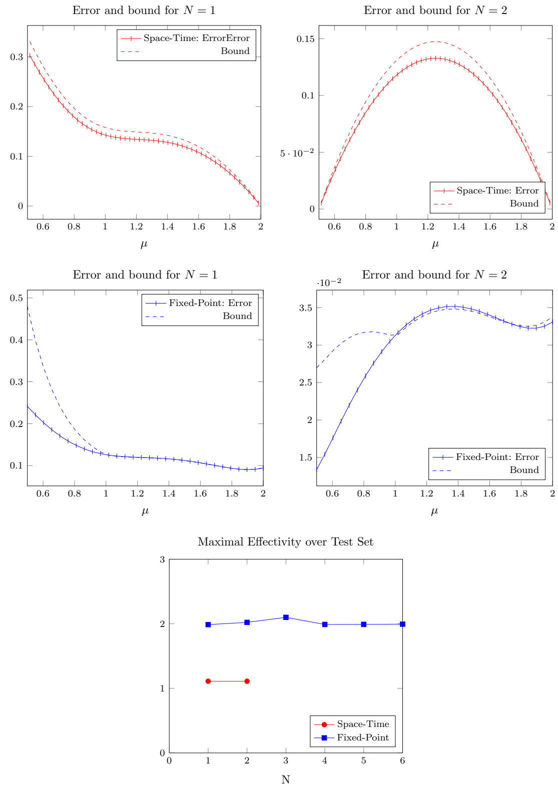


Fig. 2. Effectivity of error bounds for $N = 1, 2$ and maximum effectivities for the heat equation example.

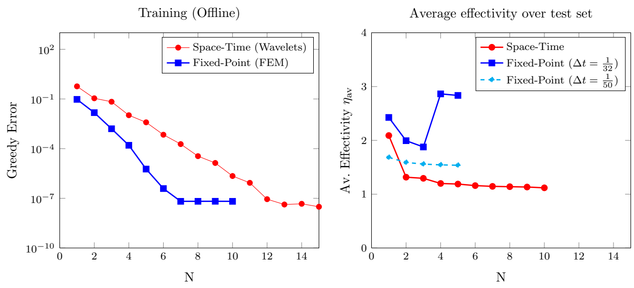
5.2 *Convection-Diffusion-Reaction equation*

For further studies, we consider a convection-diffusion-reaction example

$$u_t - u_{xx} + \mu_1 \left(\tfrac{1}{2} - x\right) u_x + \mu_2 u = g \qquad \text{on } \Omega = [0, 1],$$

with homogeneous Dirichlet boundaries in space and the time-periodicity condition $u(0) = u(T)$ for $T = 1$. As right hand side we choose $g(t, x) = \cos(2\pi t)$. The parameter set is $\mathcal{D} := [0, 30] \times [-9, 15]$. The training and test results for $|\Xi_{\text{train}}| = 400$, $|\Xi_{\text{test}}| = 225$, both sets uniformly spaced in $\mathcal{D}$, are presented below.

We see in Fig. 3 that in this example the training error $\max_{\mu\in\Xi_{\text{train}}} \Delta_N(\mu)$ decreases faster for the fixed-point method, so that the number of basis functions to reach



(a) Training (b) Average effectivity

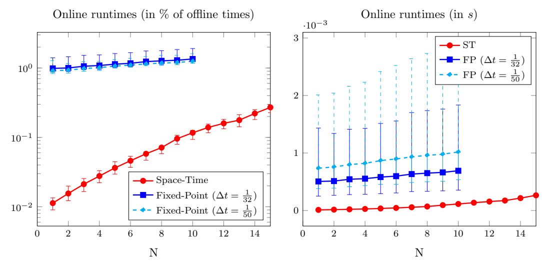
Fig. 3. Comparison of space-time and fixed-point approach for the convection-diffusion-reaction example.

a given tolerance is approximately half as large as in the space-time approach. Recall, however, that with this method we construct only a spatial reduced basis and thus have an online effort of the order of $\mathcal{O}(MKN^3)$ in contrast to $\mathcal{O}(N^3)$ in the space-time case.

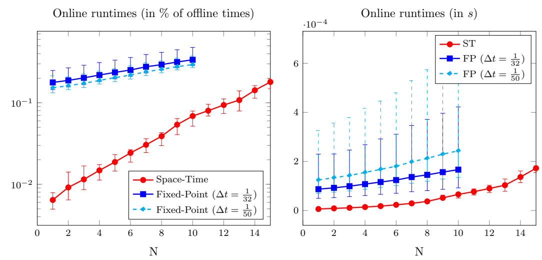
The average effectivities over the test set for different N are presented in Fig. 3b. It is apparent that the space-time error bound is consistently very good with $\eta_{av} = \frac{1}{n_{\text{test}}} \sum_{\mu\in\Xi_{\text{test}}} \eta(\mu)$ close to unity. For the fixed-point method, however, we observe that a uniformly effective bound requires a minimum number of time steps – in this example more than the corresponding number of temporal wavelets in the space-time discretization (where $\frac{1}{\Delta t} = 32$).

	Average Runtime
Space-Time	$0.0981\ s$
Fixed-Point ($\Delta t = \frac{1}{32}$)	$0.0542\ s$
Fixed-Point ($\Delta t = \frac{1}{50}$)	$0.0817\ s$

(a) Runtime comparison for truth solution u



(b) Runtime reduction (including Δ_N) (c) Online runtimes (including Δ_N)



(d) Runtime reduction (without Δ_N) (e) Online runtimes (without Δ_N)

Fig. 4. Runtime reductions for the convection-diffusion-reaction example.

To further discuss the online behaviour of both methods, we compare in Fig. 4 average runtimes [1] for a uniformly randomly chosen parameter set with $n_{\text{test}} = 25$ (error bars indicate minimum/maximum runtimes). Even though a full space-time solution here is computationally more expensive than a fixed-point solution (Fig. 4a), both the runtime reduction (reduced solution time in % of full solution time, Fig. 4b/4d) as well as the online runtimes (Fig. 4c/4e) are significantly better in space-time. This holds for online runtimes in- and excluding the computation of error bound Δ_N.

Moreover, the online runtimes of the fixed-point calculations vary significantly over the parameter set. This is caused by the different number of necessary iterations, even in this with $1 \leq M \leq 11$ rather moderate example.

6. CONCLUSION

We have proposed a space-time framework for the construction of reduced basis approximations of time-periodic solutions and compared it to an alternative approach using fixed-point time-stepping methods. The latter allow rigorous a-posteriori estimates for the approximation error in a discrete $\mathcal{Y}$-norm, whereas we proved error bounds in both $\mathcal{Y}$ and the solution space $\mathcal{X}$ in the space-time context.

Numerical experiments confirm the good effectivity of the error bounds and show that both methods yield reduced bases of similar size. This implies a significant reduction of online computational effort within the space-time framework, as there online only one equation system has to be solved ($\mathcal{O}(N^3)$) whereas the calculation in the fixed-point framework requires the computation of several initial value problems until convergence is reached ($\mathcal{O}(MKN^3)$). Moreover, the independence of the fixed-point iteration number $M = M(\mu)$ in the space-time online calculations allows a reliable a-priori prediction of online runtimes, which may be crucial for real-time problems.

Note that the space-time context allows to compute problems that are not directly feasible in a time-stepping approach, e.g. if the bilinear forms cannot be separated into time and space contributions.

The presented experiments focused on the error in the state variable. However, we showed that output errors involving a dual problem can easily be derived, even in the case of time-dependent (non LTI) operators.

Future work will also be concerned with the efficient solution of the space-time problem. Adaptive wavelet methods allow economical representations of the solution to elude the additional computational complexity introduced by the time dimension. However, first experiments indicate that the approximation error with respect to the exact solution has apparently be taken into account.

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[1] All numerical experiments have been performed on a 3.06 Ghz Intel Core Duo 2 with 4 GB RAM; runtimes are always averaged over 1000 computations.

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