Exercise Sheet 11
Due January 17th, 2017

Problem 1 (Sparsity) (5 Points)
In this exercise we consider the sparsity pattern of a stiffness matrix generated from a uniform wavelet discretization. Suppose we are given a basis of wavelets up to level $j > 0$, i.e.

$$\Psi^j := \{ \psi^j_\lambda : \lambda \in J^j \},$$

where

$$J^j := I_0 \cup J_0 \cup \ldots \cup J_{j-1},$$

with $I_0$ being the index set of scaling functions on level $j = 0$ and $J_i$ being the index set of wavelet functions on level $i$. We assume the problem is given on a bounded domain, thus, $J^j$ is finite. A wavelet index has the form $\lambda = (j,k)$.

Next, we suppose a coercive and bounded bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is given. Assume $a(\cdot, \cdot)$ is local in the sense that $a(u, v) = 0$ if $\text{supp } u \cap \text{supp } v = \emptyset$. The discrete representation of the operator w.r.t. $\Psi^j$ is given by

$$A := (a(u, v))_{u,v \in \Psi^j}.$$

Finally, denote the spanned space by $S_j := \text{span}(\Psi^j)$. We know that the stiffness matrix arising from the discretization of $a(\cdot, \cdot)$ with scaling functions is sparse. What about wavelets?

(a) Determine (with proof) the asymptotic number of nonzero entries in $A$ in dependence on $N_j := \dim(S_j)$ (i.e., $O(f(N_j)))$.

(b) Sketch the sparsity pattern of $A$.

Problem 2 (Dual Wavelets) (5 Points)
Let $v \in H$ be given by a finite wavelet expansion

$$v = \sum_{\lambda = (j,k) \in \Lambda} c_\lambda \tilde{\psi}_\lambda,$$

for $\Psi$ being a Riesz Basis for the Hilbert space $H$ and $\tilde{\Psi}$ the corresponding dual wavelets. Determine the expansion of $v$ in terms of $\Psi$.

Problem 3 (Thresholding) (5 Points)
Let $f \in L_2(\Omega)$ have the representation $f = d^T \Psi := \sum_{\lambda \in \Lambda} d_\lambda \psi_\lambda$, where $\Psi$ is a wavelet Basis. The $N$-term-thresholding of $f$ is defined as

$$T_N(f) := \sum_{\lambda \in \Lambda_N(f)} d_\lambda \psi_\lambda,$$
where $\Lambda_N(f)$ is the index set of $N$ largest wavelet coefficients $d_\lambda$ in absolute value. Denote the error of $N$-term-threshholding by

$$e_N(f) := \|f - T_N(f)\|_0.$$  

Denote the best $N$-term approximation error by

$$\rho_N(f) := \inf_{\# \text{supp } c \leq N} \|f - c^T \Psi\|_0.$$  

In which relation ($=, \leq, \geq$ etc.) is $e_N(f)$ to $\rho_N(f)$

(a) if $\Psi$ is an orthonormal basis?

(b) If $\Psi$ is a Riesz basis?