**Problem 1 (Subdivision) (20 Points)**

Recall that using a mask \( a \) we can define a subdivision operator \( S_a : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z}) \) by

\[
(S_a c)_k := \sum_{m \in \mathbb{Z}} a_{k-2m} c_m, \quad c \in \ell_\infty(\mathbb{Z}),
\]

and the corresponding subdivision scheme as

\[
c^0 := c, \quad c^m := S_a c^{m-1}, \quad m = 1, 2, \ldots
\]

We will only be considering finite masks.

The subdivision scheme is said to converge for \( c \in \ell_\infty(\mathbb{Z}) \) if there exists a function \( f_c \in C(\mathbb{R}) \) such that

\[
\lim_{m \to \infty} \| f_c(\frac{x}{2^m}) - c^m \|_{\ell_\infty(\mathbb{Z})} = 0.
\]

In this exercise we are going to show that a finite mask \( a \) determines a compactly supported refinable function and \( \varphi \) is unique given the normalization

\[
\sum_{k \in \mathbb{Z}} \varphi(x - k) = 1, \quad x \in \mathbb{R},
\]

with the limit \( f_c \) given as

\[
f_c(x) = \sum_{k \in \mathbb{Z}} c_k \varphi(2x - k), \quad x \in \mathbb{R},
\]

for all \( c \in \ell_\infty(\mathbb{Z}) \).

You will need the following Proposition.

**Proposition 1.** Suppose the subdivision scheme converges for some \( c \in \ell_\infty(\mathbb{Z}) \) and \( f_c \neq 0 \). Then the mask satisfies

\[
\sum_{m \in \mathbb{Z}} a_{k-2m} = 1, \quad k \in \mathbb{Z}.
\]

Furthermore, given a finite mask \( a \), we define an operator \( T \) by

\[
(T \varphi)(x) = \sum_{k \in \mathbb{Z}} a_k \varphi(2x - k),
\]

for any \( \varphi \in C_0(\mathbb{R}) \), i.e., for any compactly supported continuous function.

(a) Let \( \varphi \in C_0(\mathbb{R}) \), \( a \in \ell_\infty(\mathbb{Z}) \) be a finite mask and \( T \) the corresponding operator defined in (2). Show

\[
\sum_{k \in \mathbb{Z}} c_k^m \varphi(2^m x - k) = \sum_{k \in \mathbb{Z}} c_k (T^m \varphi)(x - k),
\]

for any \( c \in \ell_\infty(\mathbb{Z}) \).

**Hint:** You may want to start by showing

\[
\sum_{k \in \mathbb{Z}} c_k (T \varphi)(x - k) = \sum_{k \in \mathbb{Z}} (S_a c)_k \varphi(2x - k).
\]
(b) Suppose the subdivision scheme defined by a finite mask \( a \) converges for \( c \in \ell_\infty(\mathbb{Z}) \) to \( f_c \in C(\mathbb{R}) \) and let \( \varphi_0 \in C_0(\mathbb{R}) \) and \( \varphi_0 \geq 0 \) for all \( x \in \mathbb{R} \). Moreover, let \( \varphi_0 \) satisfy
\[
\sum_{k \in \mathbb{Z}} \varphi_0(x - k) = 1, \quad x \in \mathbb{R}.
\]
Show that for \( x \in [a, b] \) for any \( a < b, a, b \in \mathbb{R} \)
\[
f_c(x) = \lim_{m \to \infty} \sum_{k \in \mathbb{Z}} c_k(T^m \varphi_0)(x - k)
\]
uniformly in \([a, b]\).

(c) Let the subdivision scheme defined by the finite mask \( a \) converge for all \( c \in \ell_\infty(\mathbb{Z}) \) and for some \( c \in \ell_\infty(\mathbb{Z}), f_c \neq 0 \). Show that the mask \( a \) determines a unique compactly supported continuous function \( \varphi \) with the properties
\[
\varphi(x) = \sum_{k \in \mathbb{Z}} a_k \varphi(2x - k), \tag{3}
\]
\[
1 = \sum_{k \in \mathbb{Z}} \varphi(x - k), \tag{4}
\]
\[
f_c(x) = \sum_{k \in \mathbb{Z}} c_k \varphi(x - k), \tag{5}
\]
for all \( x \in \mathbb{R} \).

**Hint:** As a candidate for \( \varphi \) you might want to consider \( f_\delta \) where
\[
\delta_k = \begin{cases} 
1, & \text{if } k = 0 \\
0, & \text{otherwise.}
\end{cases}
\]

**Problem 2 (Scaling Functions)** (5 Points)
Let \( \varphi \) be the scaling function defined by a finite mask \( a \) as in the previous problem. Show

(a) \( \int_\mathbb{R} \varphi(x)dx = 1 \),

(b) \( \int_\mathbb{R} \varphi_{[j,k]}(x)dx = 2^{-j/2}, \ j \in \mathbb{N}_0, \ k \in \mathbb{Z} \),

(c) \( \|\varphi_{[j,k]}\|_0 = \|\varphi\|_0 \).

**Problem 3 (B-splines)** (5 Points)
The function \( N_d : \mathbb{R} \to \mathbb{R} \) defined recursively by
\[
N_d(x) := \int_0^1 N_{d-1}(x - t)dt, \quad N_1(x) := \chi_{[0,1]}(x)
\]
is called a *cardinal B-spline of order \( d \)*. Show

(a) \( \text{supp} N_d \subset [0, d] \),

(b) \( N_d(x) \geq 0 \) and \( N_d(x) > 0 \) for \( x \in (0, d) \),

(c) \( \int_\mathbb{R} N_d(x)dx = 1 \) and \( \sum_{k \in \mathbb{Z}} N_d(x - k) = 1 \).