1. **Theorem 1 (Nash, 1950)** Every finite normal-form static game of complete information has a mixed-strategy Nash equilibrium.

The idea of the proof of Nash’s theorem is to apply Kakutani’s fixed-point theorem to the players’ “reaction correspondences”. Let player $i$’s reaction correspondence $r_i$ map each strategy profile $\sigma$ to the set of mixed strategies that maximize player $i$’s payoff when his opponents play $\sigma_{-i}$. We define the correspondence $r : \Sigma \rightarrow \Sigma$ to be the Cartesian product of the $r_i$. A fixed point of $r$ is a $\sigma$ such that $\sigma \in r(\sigma)$, so for every player, $\sigma_i \in r_i(\sigma)$. Thus, a fixed point of $r$ is a Nash equilibrium. From Kakutani’s theorem, the following are sufficient conditions for $r : \Sigma \rightarrow \Sigma$ to have a fixed point:

- (a) $\Sigma$ is a compact, convex, nonempty subset of a finite-dimensional Euclidean space. [1 Point]
- (b) $r(\sigma)$ is nonempty for all $\sigma$. [1 Point]
- (c) $r(\sigma)$ is convex for all $\sigma$. [1 Point]
- (d) $r(\cdot)$ has a closed graph: For any two sequences $(\sigma^n), (\hat{\sigma}^n)$ with $\hat{\sigma}^n \in r(\sigma^n)$ for all $n$, $\sigma^n \rightarrow \sigma$ and $\hat{\sigma}^n \rightarrow \hat{\sigma}$, the following holds: $\hat{\sigma} \in r(\sigma)$. [3 Points]

First note that each $\Sigma_i$ is a simplex of dimension $(|\Sigma_i| - 1)$, which is nonempty, compact, and convex. So this also holds for the cartesian product $\Sigma$. Now prove that the conditions (b)-(d) are satisfied. Use the following information.

- A set $X$ in a linear vector space is convex if, for any $x$ and $y$ belonging to $X$ and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y$ belongs to $X$.
- In a game as in the theorem above, each player’s payoff function is multilinear, that is,
  \[ u_i(\lambda \sigma'_i + (1 - \lambda) \sigma''_i, \sigma_{-i}) = \lambda u_i(\sigma'_i, \sigma_{-i}) + (1 - \lambda) u_i(\sigma''_i, \sigma_{-i}) \]
  for all $\sigma'_i, \sigma''_i \in \Sigma_i$ and $\lambda \in [0, 1]$. Linear functions are continuous.
- Continuous functions on nonempty convex and compact sets attain maxima. [6 Points]
2. Solve for the mixed-strategy Nash equilibria in the following normal-form game.

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<thead>
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<th>L</th>
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<tbody>
<tr>
<td>T</td>
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<td>1,1</td>
<td>4,2</td>
</tr>
<tr>
<td>M</td>
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<td>1,2</td>
<td>2,3</td>
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<tr>
<td>B</td>
<td>1,3</td>
<td>0,2</td>
<td>3,0</td>
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[4 Points]

3. Suppose there are \( I \) farmers, each of whom has the right to graze cows on the village common. The amount of milk a cow produces depends on the number of cows, \( N \), grazing on the green. The revenue produced by \( n_i \) cows is \( n_i u(N) \) for \( N < N^* \) and \( u(N) = 0 \) for \( N \geq N^* \), where \( u(0) > 0, u' < 0 \) and \( u'' < 0 \). Each cow costs \( c \), and cows are perfectly divisible. Suppose \( u(0) > c \). Farmers simultaneously decide how many cows to purchase. All purchased cows will graze on the common. Write this game as a static game of complete information, give a pure-strategy Nash Equilibrium and compare it against the social optimum. How does this game relate to the Cournot oligopoly model?

[6 Points]

4. Consider the Cournot duopoly model from the lecture. Apply the first two steps of the iterated elimination of strictly dominated strategies: prove that the monopoly quantity \( q_m = \frac{a-c}{2} \) strictly dominates any higher quantity and that half the monopoly quantity \( \frac{q_m}{2} = \frac{a-c}{4} \) strictly dominates any lower quantity. How many steps are required in total?

[4 Points]