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Mathematics of Games

Exercise Session 5

Exercise Session 5 due on 02.06.2014, by 12:15pm, N24-H14.
Total : 20 Points
Hand-in IN PAIRS!

1. Consider a variant of Rubinstein's infinite-horizon bargaining game where partitions (x_1, x_2), with $x_1 + x_2 = 1$, are restricted to be integer multiples of 0.01, that is, x_i can be $0, 0.01, 0.02, \ldots, 0.99$, or 1 for $i = 1, 2$. There is a common discount factor δ. Prove that, if $\delta > 0.99$, all partitions can be supported as a subgame-perfect NE.

   Hint: If player i always offers (x_i, x_j), which would be the threshold value t for i to accept an offer (y_i, y_j) from j? Can player i gain by making an offer with a value lower or higher than x_i for its partition? Can player i lose by accepting any offer from j of at least t or by rejecting any offer from j less than t?

   [5 Points]

2. Consider the static Bertrand duopoly model (with homogeneous products): The two firms name prices simultaneously; demand for firm i's product is $a - p_i$ if $p_i < p_j$, is 0 if $p_i > p_j$, and is $(a - p_i)/2$ if $p_i = p_j$; marginal costs are $c < a$. Consider the infinitely repeated game based on this stage game. Assume that both players play the following strategy:

   "Play price $\frac{a+c}{2}$ in the first period. In the t^{th} period, play price $\frac{a+c}{2}$ if both played $\frac{a+c}{2}$ in each of the $t-1$ previous periods, otherwise, play price c."

   Show that the above strategy for both firms is a subgame-perfect Nash equilibrium if and only if $\delta \geq 1/2$.

   [5 Points]

3. Prove the following theorem.

Theorem 1 (Friedman 1971) *Let σ^* be a static equilibrium (an equilibrium of the stage game) with payoffs e. Then for any $v \in V$ with $v_i > e_i$ for all players i, there is a $\underline{\delta}$ such that for all $\delta > \underline{\delta}$ there is a subgame-perfect equilibrium of $G(\delta)$ with payoffs v.*

Hint: Assume initially $\exists a', u(a') = v$, and profile s where each player plays a'_i in period 0. What happens if a player deviates?

Friedman's result shows that patient and identical Cournot duopolists can implicitly collude by each producing half of the monopoly output, with any deviation triggering a switch to the Cournot outcome forever after. The collusion is implicit: Each firm is deterred from breaking the agreement by the (credible) fear of provoking Cournot's competition.

[5 Points]

4. Prove the following theorem.

Theorem 2 (Abreu 1998) *If the stage game is finite, any distribution over infinite histories that can be generated by some subgame-perfect equilibrium σ can be generated with a strategy profile σ^* that specifies that play switches to the worst equilibrium $\underline{w}(i)$ for player i if player i is the first to play an action to which σ assigns probability* 0.

Hint: Construct a profile σ^* such that $\sigma^*(h^t) = \sigma(h^t)$ as long as σ gives the history h^t positive probability. If σ gives positive probability to $h^{t'}$ for all $t' < t$, and player i is the only player to play an action with probability 0 in $\sigma(h^t)$ at period t, then play switches to the worst subgame-perfect equilibrium for player i, which is $\underline{w}(i)$. So then $\sigma^*(h^{t+1}) = \underline{w}(i)(h^0)$ and $\sigma^*((h^{t+1}, a^{t+1})) = \underline{w}(i)(a^{t+1})$, and so on.

[5 Points]