1. Consider a variant of Rubinstein’s infinite-horizon bargaining game where partitions \((x_1, x_2)\), with \(x_1 + x_2 = 1\), are restricted to be integer multiples of 0.01, that is, \(x_i\) can be 0, 0.01, 0.02, \ldots, 0.99, or 1 for \(i = 1, 2\). There is a common discount factor \(\delta\). Prove that, if \(\delta > 0.99\), any partition can be supported as a subgame-perfect NE. 

*Hint:* If player \(i\) always offers \((x_i, x_j)\), which would be the threshold value \(t\) for \(i\) to accept an offer \((y_i, y_j)\) from \(j\)? Can player \(i\) gain by making an offer with a value lower or higher than \(x_i\) for its partition? Can player \(i\) lose by accepting any offer from \(j\) of at least \(t\) or by rejecting any offer from \(j\) less than \(t\)?

2. Consider the static Bertrand duopoly model (with homogeneous products): The two firms name prices simultaneously; demand for firm \(i\)’s product is \(a - p_i\) if \(p_i < p_j\), is 0 if \(p_i > p_j\), and is \((a - p_i)/2\) if \(p_i = p_j\); marginal costs are \(c < a\). Consider the infinitely repeated game based on this stage game. Assume that both players play the following strategy: “Play price \(a + c/2\) in the first period. In the \(t^{th}\) period, play price \(a + c/2\) if both played \(a + c/2\) in each of the \(t - 1\) previous periods, otherwise, play price \(c\).”

Show that the above strategy for both firms is a subgame-perfect Nash equilibrium if and only if \(\delta \geq 1/2\).

3. Three players bargain over the division of a pie of size 1. A division is a triple \((x_1, x_2, x_3)\) of shares for each player, where \(x_i \geq 0, \sum x_i = 1\). At dates \(3k + 1, k = 0, 1, \ldots\), player 1 offers a division, then players 2 and 3 simultaneously decide whether they accept or veto. If players 2 and 3 both accept, the game is over. If one or both of them veto, bargaining goes on. Similarly, at dates \(3k + 2\) (respectively, \(3k\)), player 2 (respectively, player 3) makes the offer. The game stops once an offer by one player has been accepted by the other two players. The players have common discount factor \(\delta\).

Show that for every \(\delta \in (0, 1)\), any partition can be supported as a subgame-perfect Nash equilibrium.