Mathematics of Games

Exercise session 8

20.06.2013, 2pm-4pm, N24-H15

Hand-in before class starts.

1. **Theorem 1 (Abreu 1998)** If the stage game is finite, any distribution over infinite histories that can be generated by some subgame-perfect equilibrium $\sigma$ can be generated with a strategy profile $\sigma^*$ that specifies that play switches to the worst equilibrium $w(i)$ for player $i$ if player $i$ is the first to play an action to which $\sigma$ assigns probability 0.

   **Hint:** Consider $\sigma^*(h^t) = \sigma(h^t)$ as long as $\sigma$ gives the history $h^t$ positive probability. What happens if $\sigma$ gives positive probability to $h^t$ for all $t' < t$, and player $i$ is the only player to play an action with probability 0 in $\sigma(h^t)$ at period $t$?

2. Consider a first-price, sealed-bid auction in which the bidders’ valuations are independently and uniformly distributed on $[0, 1]$. Bidder $i$ has valuation $v_i$ for the good - that is, if bidder $i$ gets the good and pays the price $p$, then $i$’s payoff is $v_i - p$. The bids $b_i \geq 0$ are submitted simultaneously, the higher bidder wins the good and pays the bid price, the other bidders get and pay 0. In case of a tie, the winner is determined uniformly at random. Show that if there are $n$ bidders, then the strategy of bidding $\frac{n-1}{n}$ times one’s valuation is a Bayesian Nash equilibrium of this auction.

3. Consider a Cournot duopoly operating in a market with inverse demand $P(Q) = a - Q$, where $Q = q_1 + q_2$ is the aggregate quantity on the market. Both firms have total costs $c_i(q_i) = cq_i$ with a constant $c$, but demand is uncertain: it is high ($a = a_H$) with probability $\theta$ and low ($a = a_L$) with probability $1 - \theta$. So the payoff depends on $a$ and is $u_i(q_i, q_j, a) = (P(Q) - c)q_i$ for both firms. Furthermore, information is asymmetric: firm 1 knows whether demand is high or low, but firm 2 does not. All of this is common knowledge. The two firms simultaneously choose quantities. What is the pure-strategy Bayesian Nash equilibrium of this game?

4. Consider the following model of Bertrand duopoly with differentiated products. Demand for firm $i$ is $q_i(p_i, p_j) = a - p_i - b_i \cdot p_j$. Costs are zero for both firms. The sensitivity of firm $i$’s demand to firm $j$’s price is either high or low. That is, $b_i$ is either $b_H$ or $b_L$, where $b_H > b_L > 0$. For each firm, $b_i = b_H$ with probability $\theta$ and $b_i = b_L$ with probability $1 - \theta$, independent of the realization of $b_j$. Each firm knows its own $b_i$ but not its competitor’s. All of this is common knowledge. Which (four) conditions define a pure-strategy Bayesian Nash equilibrium of this game? Solve for such an equilibrium in the case $\theta = 1$. 

---

1