Proof of Hu’s theorem

This short note contains the details of the proof of Theorem 24 in [1].

**Theorem 1** (Hu [2]). Suppose there is a $c > 2$ so that any separating union-closed family $A'$ with $|A'| \leq c|U(A')|$ satisfies the union-closed sets conjecture. Then, in every union-closed family $A$, there is an element that appears in at least $\frac{c-2}{2(c-1)}|A|$ member-sets of $A$.

**Proof.** Let $A$ be a union-closed family. Clearly, we may assume $A$ to be separating. Moreover, we can suppose that $n := |A| > cm$, where $m := |U(A)|$ is the size of the universe; otherwise there is even, by assumption, an element in $U(A)$ that appears in at least half of the member-sets.

Put

$$p := \left\lceil \frac{n - cm}{c - 1} \right\rceil \leq \frac{n - cm}{c - 1} + 1 \quad (1)$$

Pick some element $x_0 \in U(A)$, and introduce $p$ new elements $X := \{x_1, \ldots, x_p\}$, disjoint from $U(A)$. We define a family

$$A' := \{A \cup X : A \in A, x_0 \in A\} \cup \{A \in A : x_0 \notin A\}$$

$$\cup \{U(A) \cup (X - x_i) : i = 1, \ldots, p\}$$

The family is obviously union-closed, and moreover, it is separating. Indeed, any elements of $U(A)$ can still be separated as $A$ is separating, while the elements $x_1, \ldots, x_p$ are separated by the sets $\{U(A) \cup (X - x_i) : i = 1, \ldots, p\}$.

The number of sets in $A'$ is $n + p$, while the universe has grown to $m + p$. We can easily check that $n + p \leq c(m + p)$, so that we can use the assumption that there is an element $u^*$ in $U(A')$ that appears in at least $\frac{n+p}{2}$ member-sets. Note that $x_0$ appears more often than any of $x_1, \ldots, x_p$, so that we may assume that $u^* \in U(A)$. Then, however, we see that $u^*$ appears in at least $\frac{n+p}{2} - p = \frac{n-p}{2}$ of the members-sets of $A$.

We compute with (1) that

$$\frac{n-p}{2} \geq \frac{1}{2} \left( \frac{n - cm}{c - 1} - 1 \right) = \frac{1}{2} n \left( \frac{c - 2}{c - 1} + \frac{1}{n} \left( \frac{cm}{c - 1} - 1 \right) \right)$$

$$> \frac{1}{2} n \cdot \frac{c - 2}{c - 1}$$

as $c > 2$ entails that $cm > c - 1$. Thus, there is an element that appears in at least $\frac{c-2}{2(c-1)}n$ of members-sets of $A$. \qed

**References**
