Bicycles and left-right tours in locally finite graphs

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joint with S. Kosuch & M. Win Myint
Bicycles in finite graphs

cycle space $\mathcal{C}(G)$

cut space $\mathcal{C}^*(G)$

bicycle space $\mathcal{B}(G) = \mathcal{C}(G) \cap \mathcal{C}^*(G)$
Tripartition theorem

Theorem (Read & Rosenstiehl)

Let $G$ finite, $e$ edge of $G$. Then exactly one of the following holds:

(i) $\exists B \in B(G)$ with $e \in B$
(ii) $\exists Y \in C(G)$ with $e \in Y$ and $Y + e \in C^*(G)$
(iii) $\exists Z \in C(G)$ with $e \notin Z$ and $Z + e \in C^*(G)$. 
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(iii) $\exists Z \in \mathcal{C}(G)$ with $e \not\in Z$ and $Z + e \in \mathcal{C}^*(G)$. 

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Tripartition in $\infty$ graphs?

- no finite bicycle $\ni e$
- no finite $Z \in C(G)$ with $Z + e \in C^*(G)$
Tripartition in $\infty$ graphs?

- no finite bicycle $\ni e$
- no finite $Z \in \mathcal{C}(G)$ with $Z + e \in \mathcal{C}^*(G)$

$\Rightarrow$ infinite bicycle $\ni e$
Hamburg version

Theorem

Let $G$ locally finite, $e$ edge of $G$. Then exactly one of the following holds:

(i) $\exists B \in \mathcal{B}(G)$ with $e \in B$

(ii) $\exists \text{finite } Y \in \mathcal{C}(G)$ with $e \in Y$ and $Y + e \in \mathcal{C}^*(G)$

(iii) $\exists \text{finite } Z \in \mathcal{C}(G)$ with $e \notin Z$ and $Z + e \in \mathcal{C}^*(G)$.

Waterloo version

Theorem (Casteels&Richter)

Let $G$ locally finite, $e$ edge of $G$. Then exactly one of the following holds:

(i) $\exists \text{finite } B \in \mathcal{B}(G)$ with $e \in B$

(ii) $\exists X \in \mathcal{C}(G)$ with $X + e \in \mathcal{C}^*(G)$

real tripartition

consequence of more general theorem
Ambiguous edges

\[ \exists Y \in \mathcal{C}(G) \text{ with } e \in Y \text{ and } Y + e \in \mathcal{C}^*(G) \]

\[ \exists Z \in \mathcal{C}(G) \text{ with } e \notin Z \text{ and } Z + e \in \mathcal{C}^*(G) \]
Property

**Locally finite** $G$ is pedestrian iff for all $e \in E(G)$ there exist finite $Z \in C(G)$ with $Z + e \in C^*(G)$.

Theorem (Read & Rosenstiehl)

$G$ finite and connected. Then $G$ pedestrian iff

$\#$ of spanning trees $= \text{odd}$. 

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Question: when is an $\infty$ graph pedestrian?
A left-right tour... 

...and its residue

**Theorem (Shank)**

*The residue of a left-right tour in a finite plane graph is a bicycle.*
A left-right tour...

...and its residue

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The residue of a left-right tour in a finite plane graph is a bicycle.
Left-right tours

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**Theorem (Shank)**

*The residue of a left-right tour in a finite plane graph is a bicycle.*

**Theorem (Horton, Shank)**

*The residues of left-right tours generate $\mathcal{B}(G)$ in finite plane $G$.*
How to define LRT in $\infty$ graphs?

Left-right tour should be...
How to define LRT in $\infty$ graphs?

Left-right tour should be...
How to define LRT in $\infty$ graphs?

Left-right tour should be...

- ...left-right
- ...a tour
- residue is bicycle
DEF of left-right tours

DEF A left-right tour is $\tau : S^1 \rightarrow |G|$ that is...

- locally left-right
- locally injective at edges

parity information is lost in ends
Results

Lemma
The residue of a left-right tour in a locally finite plane graph is a bicycle.

Theorem
The residues of left-right tours generate $B(G)$ in locally finite plane $G$.

Problems:

- Finite proof uses plane duals
- Existence of left-right tours?
Unique LRT in pedestrians

- pedestrian graph has unique LRT
- planarity criterion lists its properties
Unique LRT in pedestrians

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**Read&Rosenstiehl’s planarity criterion**

**DEF** halves

**DEF** tour $W$ is algebraic diagonal if
- double cover
- each residue of a half is a cut

**Theorem (Read&Rosenstiehl)**

*Let $G$ be finite and pedestrian. Then $G$ is planar iff it has algebraic diagonal.*

→ extends to locally finite
→ for non-pedestrian graphs: Archdeacon, Bonnington & Little
**Read & Rosenstiehl’s planarity criterion**

**DEF** halves

![Diagram of halves]

**DEF** tour $W$ is algebraic diagonal if

- double cover
- each residue of a half is a cut

**Theorem (Read & Rosenstiehl)**

*Let $G$ be finite and pedestrian. Then $G$ is planar iff it has algebraic diagonal.*

→ extends to locally finite
→ for non-pedestrian graphs:
Archdeacon, Bonnington & Little
Open question

- characterise pedestrian graphs
- left-right tours on other surfaces
- left-right tours for other compactifications