The union-closed sets conjecture

Henning Bruhn

joint with Oliver Schaudt
The union-closed sets conjecture

Always: \( \mathcal{A} \) finite family of finite sets

- \( \mathcal{A} \) union-closed: \( A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \).
- Example: \( \emptyset, 1, 12, 34, 134, 1234 \)

Conjecture

Every union-closed family of at least two sets has an element that appears in at least half of the member-sets.

- power sets are union-closed
- conjecture tight for power sets!
The union-closed sets conjecture

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- \( \mathcal{A} \) union-closed: \( A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \).
- Example: \( \emptyset, 1, 12, 34, 134, 1234 \)

**Conjecture**

*Every union-closed family of at least two sets has an element that appears in at least half of the member-sets.*

- power sets are union-closed
- conjecture tight for power sets!
A union-closed family

- union-closed
- 25 sets
- **Universe**: 1, 2, 3, 4, 5, 6
A union-closed family

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- union-closed
- 25 sets
- **Universe:** 1, 2, 3, 4, 5, 6
- 2 lies in 12 member-sets
A union-closed family

- union-closed
- 25 sets
- **Universe**: 1, 2, 3, 4, 5, 6
- 2 lies in 12 member-sets
- 4 lies in 16 member-sets → conjecture ✔
Peter Winkler ’87:

*The ‘union-closed sets conjecture’ is well known indeed, except for (1) its origin and (2) its answer!*
Some terminology

Conjecture

Every union-closed family of at least two sets has an element that appears in at least half of the member-sets.

- $\mathcal{A}$ always (finite) union-closed family
- $U := \bigcup_{A \in \mathcal{A}} A$ is the universe
- frequency: $\mathcal{A}_u := \{ A \in \mathcal{A} : u \in A \}$
- $u$ abundant if $|\mathcal{A}_u| \geq \frac{1}{2} |\mathcal{A}|$
- $n$: number of member-sets
- $m$: number of elements
What do we know?

\( \mathcal{A} \) satisfies the conjecture when
- at most 12 elements
- at most 50 member-sets
- \( \mathcal{A} \) has special structure, for instance represented by cubic graph
- ...

Also
- some (weak) properties of smallest counterexample known

\[ \rightarrow \text{conjecture wide open} \]
Knill’s argument

Knill: There’s always an element appearing in $\geq \frac{n-1}{\log_2(n)}$ member-sets

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$X :$ smallest set intersecting all members

traces $\mathcal{T} = \{A \cap X : A \in \mathcal{A}\}$

$X = 12456?$

smallest traces $12|4|5|6$
Knill’s argument

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$X$ : smallest set intersecting all members
traces $\mathcal{T} = \{A \cap X : A \in \mathcal{A}\}$

- $X = 1456$
- $\mathcal{T}$ contains all singletons of $X$
- union-closed $\Rightarrow \mathcal{T} = 2^X$
- $|X| = \log_2(|\mathcal{T}|) \leq \log_2(n)$
- an element in $X$ meets at least $(n - 1)/\log_2(n)$ member-sets
Knill’s argument

Knill: There’s always an element appearing in \( \geq \frac{n-1}{\log_2(n)} \) member-sets.

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12345 & 12346 & 12356 \\
12456 & 13456 & 23456 \\
1234 & 1235 & 1236 \\
1456 & 2456 & 3456 \\
123 & 145 & 246 & 356 & 456 \\
45 & 46 & 56 \\
4 & 5 & 6 \\
\emptyset
\end{array}
\]

\( X \): smallest set intersecting all members traces \( T = \{ A \cap X : A \in A \} \)

- \( X = 1456 \)
- \( T \) contains all singletons of \( X \)
- union-closed \( \rightarrow T = 2^X \)
- \( \rightarrow |X| = \log_2(|T|) \leq \log_2(n) \)
- \( \rightarrow \) an element in \( X \) meets at least \( (n - 1)/\log_2(n) \) member-sets

- constant factor improved by Wójcik
Many faces

Equivalent reformulations

- in terms of lattices
- in terms of graphs
- in terms of “very full sets”
Lattice: Finite poset \((L, <)\) so that

- any two \(x, y \in L\) have unique greatest lower bound \(x \wedge y\)
- any two \(x, y \in L\) have unique smallest upper bound \(x \vee y\)
- non-zero \(x \in L\) is join-irreducible if \(x = y \vee z\) implies \(x = y\) or \(x = z\).
Lattice: Finite poset \((L, <)\) so that

- any two \(x, y \in L\) have unique greatest lower bound \(x \land y\)
- any two \(x, y \in L\) have unique smallest upper bound \(x \lor y\)

non-zero \(x \in L\) is join-irreducible if \(x = y \lor z\) implies \(x = y\) or \(x = z\).
Conjecture

Let $L$ be a lattice with $|L| \geq 2$. Then there is join-irreducible $x \in L$ so that

$$|\{y : x \leq y\}| \leq \frac{1}{2}|L|.$$
Main techniques

- Injections
- Local configurations
- Averaging
Injections

**Up-set:** If $A \in \mathcal{A}$ and $B \supseteq A$ then $B \in \mathcal{A}$

- Up-sets satisfy the conjecture

**Proof:**
- Injection $\mathcal{A}_x \rightarrow \mathcal{A}_x$
- $\Rightarrow 2|\mathcal{A}_x| \geq |\mathcal{A}_x| + |\mathcal{A}_x| = |\mathcal{A}|$
- $x$ abundant!

**Problem with this technique:**
Need to know where to find an abundant element
Local configurations

Early observation:

*Singleton*: \( \{ x \} \in \mathcal{A} \rightarrow x \) abundant!

*2-set*: \( \{ x, y \} \in \mathcal{A} \rightarrow x \) or \( y \) abundant!

*3-set*: \( \{ x, y, z \} \in \mathcal{A} \rightarrow x, y \) or \( z \) abundant?

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\( \emptyset \)
Early observation:

Singleton \( \{ x \} \in \mathcal{A} \) \( \longrightarrow \) \( x \) abundant!

2-set \( \{ x, y \} \in \mathcal{A} \) \( \longrightarrow \) \( x \) or \( y \) abundant!

3-set \( \{ x, y, z \} \in \mathcal{A} \) \( \longrightarrow \) \( x, y \) or \( z \) abundant?

NO!

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However:

- $123, 124, 134 \in \mathcal{A}$ then one of $1, 2, 3, 4$ is abundant
- Family $\mathcal{L}$ is Frankl-complete if any union-closed $\mathcal{A}$ that contains $\mathcal{L}$ has abundant element among the elements of $\mathcal{L}$
- all Frankl-complete families known on five elements (Morris)
- General characterisation due to Poonen
### Averaging

Strategy: determine **average frequency**

\[
\frac{1}{|U|} \sum_{u \in U} |A_u|
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- 1, 2, 3 → each 12 times
- 4, 5, 6 → each 16 times
- average frequency
  \[
  \frac{1}{6} (3 \cdot 12 + 3 \cdot 16) = 15
  \]
- → there is element of frequency ≥ 15
- 25 member-sets → ✔
Averaging

Strategy: determine average frequency \( \frac{1}{|U|} \sum_{u \in U} |\mathcal{A}_u| \)

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- \( \emptyset \)

- 1, 2, 3 \( \rightarrow \) each 12 times
- 4, 5, 6 \( \rightarrow \) each 16 times
- average frequency \( \frac{1}{6}(3 \cdot 12 + 3 \cdot 16) = 15 \)
- \( \rightarrow \) there is element of frequency \( \geq 15 \)
- 25 member-sets \( \rightarrow \) ✔

complete rubbish approach!
Double counting:

\[ \sum_{A \in \mathcal{A}} |A| = \sum_{u \in U} |\mathcal{A}_u| \]

Usually, total set size easier to control!
Thus, if

\[ \frac{1}{|U|} \sum_{A \in \mathcal{A}} |A| \geq \frac{1}{2} |\mathcal{A}| \]

then \( \mathcal{A} \) satisfies the conjecture.
Double counting:
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Usually, total set size easier to control!
Thus, if average set size
\[ \frac{1}{|\mathcal{A}|} \sum_{A \in \mathcal{A}} |A| \geq \frac{1}{2} |U| \]
then \( \mathcal{A} \) satisfies the conjecture.

- Advantage: Don’t need to know where to look for abundant element
- Drawback: Averaging does not always work!
Large families

Nishimura & Takahashi ’96:

- If $|\mathcal{A}| > 2^m - \sqrt{2^m}$, where $m = |U|$ then $\mathcal{A}$ ✓

Proof:

Subfamily of power set on 1, 2, 3, 4
Large families

Nishimura & Takahashi ’96:

- If $|\mathcal{A}| > 2^m - \sqrt{2^m}$, where $m = |U|$ then $\mathcal{A}$

Proof:

- Assume set $X \notin \mathcal{A}$ with $|X| \geq \frac{m}{2}$
- If $Y \subseteq X$ in $\mathcal{A} \Rightarrow Y \setminus X \notin \mathcal{A}$
- Thus: $\frac{1}{2}2^{|X|}$ sets missing in $\mathcal{A}$
- $|\mathcal{A}| \leq 2^m - 2^{\frac{m}{2}}$, contradiction!

Subfamily of power set on 1, 2, 3, 4
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- $A$ contains all large sets
- Average set size $\geq \frac{m}{2}$

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- $|\mathcal{A}| \leq 2^m - 2^{\frac{m}{2}}$, contradiction!
- ⇒ $\mathcal{A}$ contains all large sets
- Average set size $\geq \frac{m}{2}$

→ $\mathcal{A}$ satisfies the conjecture!
Balla, Bollobás & Eccles ’13:
- If $|\mathcal{A}| \geq \frac{2}{3}2^m$, where $m = |U|$ then $\mathcal{A}$

Result...
- is best possible
- builds on Kruskal-Katona theorem
- and on approach of Reimer

Reimer ’03:
- Average set size always

$$\frac{1}{|\mathcal{A}|} \sum_{A \in \mathcal{A}} |A| \geq \frac{1}{2} \log_2 |\mathcal{A}|$$
Let $A, B \subset \mathbb{N}$ finite
Set $A < B$ if

1. largest element: $\max A < \max B$
2. reverse colex: $\max(A \Delta B) \in A$

Initial segment:

\[
\emptyset < 1 < 12 < 2 < 123 < 23 < 13 < 3 < 1234 < 234 < 134 < 34 < 124 < 24 < 14 < 4 < 12345 < \ldots
\]

Czédli, Maróti and Schmidt ’09:

- $\mathcal{H}(m)$: initial segment of length $\left\lfloor \frac{2}{3} 2^m \right\rfloor$
- Average too low for $\mathcal{H}(m)$!
Consider $\mathcal{A} = \emptyset, 1, 12, 234, 1234$

- 4 does not add anything!
- delete 4 from every member
- $\mathcal{A}' = \emptyset, 1, 12, 23, 123$

$\rightarrow$ may assume that $\mathcal{A}$ separating:

for every $x, y \in U$ there is $A \in \mathcal{A}$ containing exactly one of $x, y$
Small families

Let $\mathcal{A}$ be separating, $U = \{x_1, \ldots, x_m\}$.

Assume $x_1, \ldots, x_m$ ordered by increasing frequency

- let $X_0$ be universe
- for $i < j$ there exists $X_{ij}$ with $x_i \not\in X_{ij} \ni x_j$
- set $X_i := \bigcup_j X_{ij}$
- all $X_0, \ldots, X_{m-1}$ distinct
- all contain $x_m$

$\rightarrow$ If $|\mathcal{A}| \leq 2m$ then $\mathcal{A}$ satisfies conjecture (Falgas-Ravry ’11)
Let $\mathcal{A}$ be separating, $U = \{x_1, \ldots, x_m\}$.

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$\rightarrow$ If $|\mathcal{A}| \leq 2m$ then $\mathcal{A}$ satisfies conjecture (Falgas-Ravry ’11)
Future directions?

- What families on $n$ member-sets have lowest max frequency?
- for $n = 2^m \rightarrow$ power sets
- in between powers of two?
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- for $n = 2^m \rightarrow$ power sets
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Details, bibliography and more: *The journey of the union-closed sets conjecture*, Henning Bruhn and Oliver Schaudt