Quadrangulations of the projective plane are $t$-perfect if and only if they are bipartite

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Abstract

We show that every non-bipartite quadrangulation of the projective plane contains an odd wheel as a $t$-minor and is thus $t$-imperfect.

1 Introduction

Perfect graphs received considerous attention in graph theory. The purely combinatorial definition — that a graph is perfect if and only if for each of its induced subgraphs, the chromatic number and the clique number coincide — can be replaced by a polyhedral one (Chvátal [6]): A graph $G$ is perfect if and only if its stable set polytope $ SSP(G)$, ie the convex hull of stable sets of $G$, is completely described by non-negativity and clique inequalities.

Modification of the considered inequalities leads to generalisations of perfection. A graph $G$ is $t$-perfect if $ SSP(G)$ equals the polyhedron $ TSTAB(G)$ which is defined via non-negativity-, edge- and odd-cycle inequalities.

As bipartite graphs are easily seen to be perfect, the above polyhedral conditions directly show that bipartite graphs are $t$-perfect as well. We show that non-bipartite quadrangulations of the projective plane are never $t$-perfect.

Corollary 1. A quadrangulation of the projective plane is $t$-perfect if and only if it is bipartite.

This directly follows from the fact that every non-bipartite quadrangulation of the projective plane has an odd wheel (see Figure 1) as a $t$-minor. Any graph obtained by a sequence of vertex deletions and $t$-contraction is a $t$-minor. A $t$-contraction, which is only allowed at a vertex with stable neighbourhood, contracts all the incident edges. See next section for a proper definition.

Theorem 2. A quadrangulation $G$ of the projective plane contains an odd wheel $W_{2k+1}$ for $k \geq 1$ as a $t$-minor if and only if $G$ is not bipartite.

Figure 1: The odd wheels $W_3, W_5$ and $W_7$

A general treatment on $t$-perfect graphs may be found in Grötschel, Lovász and Schrijver [12, Ch. 9.1] as well as in Schrijver [14, Ch. 68]. Bruhn and
Benchetrit showed that plane triangulations are $t$-perfect if and only if they do not contain a certain subdivision of an odd wheel as an induced subgraph \[3\].

Boulala and Uhry \[4\] established the $t$-perfection of series-parallel graphs. Gerards \[9\] extended this to graphs that do not contain an odd-$K_4$ as a subgraph (an odd-$K_4$ is a subdivision of $K_4$ in which every triangle becomes an odd circuit). Gerards and Shepherd \[10\] characterised the graphs with all subgraphs $t$-perfect, while Barahona and Mahjoub \[2\] described the $t$-imperfect subdivisions of $K_4$. Bruhn and Fuchs \[5\] characterised $P_5$-free and near-bipartite graphs by forbidden $t$-minors.

Youngs \[15\] showed that all non-bipartite quadrangulations of the projective plane have chromatic number equal to 4. Esperet and Stehlík \[8\] gave bounds for edge- and face-width of non-bipartite quadrangulations.

## 2 $t$-perfection

All the graphs mentioned here are finite and simple; we follow the notation of Diestel \[7\].

Let $G = (V, E)$ be a graph. The **stable set polytope** $\text{SSP}(G) \subseteq \mathbb{R}^V$ of $G$ is defined as the convex hull of the characteristic vectors of stable, ie independent, subsets of $V$. The characteristic vector of a subset $S$ of the set $V$ is the vector $\chi_S \in \{0, 1\}^V$ with $\chi_S(v) = 1$ if $v \in S$ and 0 otherwise. We define a second polytope $\text{TSTAB}(G) \subseteq \mathbb{R}^V$ for $G$, given by

\[
x \geq 0, \\
x_u + x_v \leq 1 \text{ for every edge } uv \in E, \\
\sum_{v \in V(C)} x_v \leq \left\lfloor \frac{|C|}{2} \right\rfloor \text{ for every induced odd cycle } C \text{ in } G.
\]

These inequalities are respectively known as non-negativity, edge and odd-cycle inequalities. Clearly, $\text{SSP}(G) \subseteq \text{TSTAB}(G)$.

The graph $G$ is called **$t$-perfect** if $\text{SSP}(G)$ and $\text{TSTAB}(G)$ coincide. Equivalently, $G$ is $t$-perfect if and only if $\text{TSTAB}(G)$ is an integral polytope, ie if all its vertices are integral vectors.

It is easy to verify that vertex deletion preserves $t$-perfection. Another operation that keeps $t$-perfection was found by Gerards and Shepherd \[10\]: whenever there is a vertex $v$, so that its neighbourhood is stable, we may contract all edges incident with $v$ simultaneously. We will call this operation a $t$-contraction at $v$. Any graph that is obtained from $G$ by a sequence of vertex deletions and $t$-contractions is a $t$-minor of $G$. Let us point out that any $t$-minor of a $t$-perfect graph is again $t$-perfect.

A **$p$-wheel** $W_p$ is a graph consisting of a cycle $(w_1, \ldots, w_p)$ and a vertex $v$ adjacent to $w_i$ for $i = 1, \ldots, p$ (see Figure 1). Odd wheels $W_{2k+1}$ for $k \geq 1$ are $t$-imperfect. Indeed, the vector $(1/3, \ldots, 1/3)$ is contained in $\text{TSTAB}(W_{2k+1})$ but not in $\text{SSP}(W_{2k+1})$. Furthermore, every proper $t$-minor of an odd wheel is $t$-perfect.
3 Embeddings in the projective plane

We begin by recalling several useful definitions related to surface-embedded graphs. For further background on topological graph theory, we refer the reader to Gross and Tucker [11] or Mohar and Thomassen [13].

An embedding of a simple graph \( G \) in the projective plane is a continuous one-to-one function from \( G \) into the projective plane where \( G \) is assumed to have the natural topology as a 1-dimensional CW-complex. For our purpose, it is convenient to abuse the terminology by referring to the image of \( G \) as the embedded graph \( G \). The connected components of the complement of an embedded graph are called the faces of \( G \). An embedding is a cell embedding if each face is homeomorphic to an open disk. A cell embedding is a closed-cell embedding if each face is bounded by a cycle of \( G \).

The size of a face is the length of its bounding cycle. An embedding \( G \) is even if all faces are of even size. A quadrangulation is a cell embedding where all faces are of size 4. Note that a quadrangulation of a simple graph \( G \) is always a closed-cell embedding as \( G \) does not contain multiple edges.

A cycle \( C \) in the projective plane is contractible if \( C \) separates the projective plane into two sets \( S_C \) and \( S_C^c \) where \( S_C \) is homeomorphic to an open disk in \( \mathbb{R}^2 \). We call \( S_C \) the interior of \( C \).

A quadrangulation is nice, if the interior of every contractible 4-cycle contains no vertex.

Since the projective plane has Euler characteristic 1, every quadrangulation \( G \) satisfies
\[
\sum_{v \in V(G)} \deg(v) = 2|E(G)| = 4|V(G)| - 4. \tag{1}
\]

A cycle in a non-bipartite quadrangulation of the projective plane is contractible if and only if it has even length (see e.g. [1, Lemma 3.1]). One can easily generalise this result.

**Lemma 3.** A cycle in a non-bipartite even embedding in the projective plane is contractible if and only it has even length.

Note that therefore any two odd cycles have an odd number of crossings.

**Proof of Lemma 3.** Let \( G \) be an even embedding. Construct a quadrangulation \( G' \) from \( G \) as follows: We quadrangulate every face bounded by edges \( e_1, \ldots, e_k \) with \( k \geq 6 \) as follows: Let \( e_i = v_i v_{i+1} \) for \( i \in [k] \) and \( v_1 = v_{k+1} \). Note that \( v_i = v_j \) for some \( i \neq j \) is possible if \( G \) is not a closed-cell embedding. Add vertices \( u, w_1 = w_{k+1}, w_2, \ldots, w_k \) and connect \( w_i \) to \( v_i \) and \( w_{i+1} \) for \( i \in [k] \). Further, add the edge \( w_i u \) for every even \( i \in [k] \).

As all cycles of \( G \) are contained in the constructed quadrangulation, we are done by [1, Lemma 3.1]. \( \square \)

Let us continue with some useful operations.

**Lemma 4.** A graph obtained from an even embedding in the projective plane by a deletion of a vertex or an edge is again an even embedding.

**Proof.** Deletion of a vertex merges all adjacent even faces to a new even face and leaves all other faces untouched.
Deletion of an edge with two adjacent faces $F_1, F_2$ merges $F_1$ and $F_2$ into a new face of size $|F_1| + |F_2| - 2$. Deletion of an edge bounding a face $F$ from two sides leads to a new face of size $|F| - 2$. In both cases, all other faces are left untouched. Therefore, even embeddability is preserved under the deletion of a vertex or an edge.

**Lemma 5.** A graph obtained from a non-bipartite nice quadrangulation $G$ by a $t$-contraction is a non-bipartite quadrangulation.

**Proof.** A $t$-contraction preserves the parity of all cycles of length at least 5 and does not create new cycles. The operation leaves every 4-cycle untouched or transforms it into an edge. As the 4-cycles in a nice quadrangulation are precisely the cycles bounding faces (Lemma 3), the graph obtained from a $t$-contraction is again a quadrangulation.

Since no odd cycle gets destroyed, the graph remains non-bipartite.

After each $t$-contraction, we can delete all vertices in the interior of a 4-cycle to obtain a nice quadrangulation. This does not destroy non-bipartiteness.

Application of $t$-contractions (and deletion of vertices in the interior of 4-cycles) gives a graph where each vertex is contained in a triangle. The following lemma is a direct consequence of this observation:

**Lemma 6.** Let $G$ be a non-bipartite quadrangulation. Then there is a sequence of $t$-contractions and deletions of vertices in the interior of 4-cycles that transforms $G$ into a non-bipartite nice quadrangulation where each vertex is contained in a triangle.

**Lemma 7.** Every nice quadrangulation has minimal degree 3.

**Proof.** Let $G$ be a quadrangulation. By (1) and as $G$ is 2-connected, $G$ contains a vertex of degree 2 or 3.

Assume that vertex $v$ has exactly two neighbours $u, u'$. Then, there are vertices $s, t$ such that $(u, v, u', s)$ and $(u, v, u', t)$ are bounding a face (Figure 2). Consequently, $(u, s, u', t)$ is a contractible 4-cycle which is not the boundary of a face and $G$ is not a nice quadrangulation.

![Figure 2: A vertex of degree 2 in a quadrangulation](image)

Finally, we consider wheels in the projective plane. *Odd wheels*, ie $p$-wheels where $p \geq 3$ is odd, are evenly embeddable in the projective plane; see Figure 3 for an illustration.

**Lemma 8.** Even wheels $W_{2k}$ for $k \geq 2$ do not have an even embedding in the projective plane.
Figure 3: An even embedding of $W_5$ and the only even embedding of $W_4 - w_3w_4$

Proof. We first consider the 4-wheel $W_4$. As all triangles of $W_4 - w_3w_4$ must be non-contractible by Lemma 3, the graph must be embedded as in Figure 3. Since the insertion of $w_3w_4$ will create an odd face, $W_4$ is not evenly embeddable.

Now assume that $W_{2k}$ for $k \geq 3$ is evenly embedded. Deleting the edges $vw_i$ for $i = 5, \ldots, 2k$ preserves the even embedding (Lemma 4). Replacing the induced odd path $w_4, w_5, \ldots, w_{2k}, w_1$ by the edge $w_4w_1$ (by $t$-contraction at the vertices $w_5, w_7, \ldots, w_{2k-1}$) yields an even embedding of $W_4$. This is not possible.

4 Proof of Theorem 2

This section is dedicated to the proof of Theorem 2 (and of Corollary 1). The theorem is based on the following lemma.

Lemma 9. Let $G$ be a non-bipartite nice quadrangulation of the projective plane where each vertex of $G$ is contained in a triangle. Then $G$ contains an odd wheel as an (induced) subgraph.

Proof. By Lemma 7 there exists a vertex $v$ of degree 3 in $G$. Let $\{x_1, x_2, x_3\}$ be its neighbourhood and let $x_1x_2$ and $v$ form a triangle.

Recall that each two triangles intersect (see Lemma 3). As $x_3$ is contained in a triangle intersecting the triangle $(v, x_1, x_2)$ and as $v$ has no further neighbour, we can suppose without loss of generality that $x_3$ is adjacent to $x_1$. The graph induced by the two triangles $(v, x_1, x_2)$ and $(x_1, v, x_3)$ is not a quadrangulation. Further, addition of the edge $x_2x_3$ yields a $K_4$. Thus, the graph contains a further vertex and this vertex is contained in a further triangle $T$. Since the vertex $v$ has degree 3, it is not contained in $T$. If further $x_1 \notin V(T)$, then the vertices $x_2$ and $x_3$ must be contained in $T$. But then $x_2x_3 \in E(G)$ and, as above, $v, x_1, x_2$ and $x_3$ form a $K_4$. Therefore, $x_1$ is contained in $T$ and consequently in every triangle of $G$. Since every vertex is contained in a triangle, $x_1$ must be adjacent to all vertices of $G' = G - x_1$. As $|E(G)| = 2|V(G)| - 2$ by (1), the graph $G'$ has $2|V(G)| - 2 - (|V(G)| - 1) = |V(G)| - 1 = |V(G')|$ many edges. Thus, $G'$ contains an induced cycle $C$ and $G$ contains a wheel formed by $C$ and $x_1$ as an induced subgraph. Since even wheels are not evenly embeddable (Lemma 8), the wheel is odd by Lemma 4.

Finally, from Lemma 9 and Lemma 6 our main result follows:
Proof of Theorem 2. As \( t \)-contraction preserves the parity of cycles, no bipartite graph has an odd wheel as a \( t \)-minor.

Let \( G \) be a non-bipartite quadrangulation. By Lemma 6, \( G \) has a non-bipartite nice quadrangulation \( G' \) where each vertex is contained in a triangle as a \( t \)-minor. By Lemma 9, \( G' \) contains an induced odd wheel. Therefore, \( G \) has an odd wheel as a \( t \)-minor.

Proof of Corollary 1. Let \( G \) be a quadrangulation of the projective plane. If \( G \) is bipartite, \( G \) is also \( t \)-perfect. Otherwise, \( G \) contains an odd wheel as a \( t \)-minor (Theorem 2). Since odd wheels are \( t \)-imperfect, \( G \) is \( t \)-imperfect as well.

References


