Covers of the affine line in positive characteristic with prescribed ramification

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Introduction

Let k be an algebraically closed field of characteristic $p > 0$. In this note we consider Galois covers $g : Y \to \mathbb{P}^1_k$ which are only branched at $t = \infty$. We call such covers *unramified covers of the affine line*. It follows from Abhyankar's conjecture for the affine line proved by Raynaud ([7]) that such a cover exists for a given group G if and only if G is a quasi-p group, i.e. can be generated by elements of p-power order. Many groups satisfy this property, for example all simple groups of order divisible by p are quasi-p groups. For a given quasi-p group G there are many unramified covers of the affine line. For example, if $G = \mathbb{Z}/p\mathbb{Z}$ there are infinitely many families of such covers.

In this note we fix a quasi-p group G and ask what the minimal genus of an unramified G-Galois cover of the affine line is. This question has been motivated by the work of Muskat–Pries ([6]). Muskat and Pries compute the genus of many alternating-group covers which had been found by Abhyankar ([1]). It turns out that the covers they consider have "small" genus in a sense which can be made precise. We show that for $p + 2 \leq d < 2p$ the covers of [6] are indeed the unramified A_d-covers of the affine line of minimal genus.

Another motivation for studying the existence of covers with given ramification comes from the theory of stable reduction. One method for showing that covers with given tame ramification exist in positive characteristic is to show that not all covers have bad reduction to characteristic p. Associated with a cover with bad reduction is a set of tail covers. Essentially these are the restriction to certain irreducible components of the stable reduction of the cover. A technique due to Wewers ([9]) sometimes allows one to count the number of covers with bad reduction in terms of sufficient knowledge of these tail covers. This technique has been applied in [4] to covers of degree p which are tamely branched at 4 points. To generalize the result of [4] to degree $p < d < 2p$ one needs to know the existence of covers of the affine line of degree d of small genus.

The strategy of the proof of our main result is as follows. Rather than considering covers with fixed Galois group, we consider unramified covers $f : X \to \mathbb{P}^1$ of the affine line which are non-Galois. Let d be the degree of f. As an application of the Riemann–Hurwitz formula in positive characteristic, we compute the genus of the Galois closure of f in terms of the ramification of f

(Proposition 1.3). We conclude that for $p+2 \leq d < 2p$ the covers considered in [6] indeed have minimal genus. For $d = p+1$ the cover of minimal genus is another well-known cover. This cover is in fact given by the same equation as the covers in degree $d \geq p+2$, but the Galois group of the Galois closure is $\mathrm{PSL}_2(p)$ (Proposition 2.1). For $d = p$, we use the theory of stable reduction to construct covers with minimal genus. In this case, we were not able to specify the Galois group G of the Galois closure.

In this paper, we always suppose that $d < 2p$. This implies that p strictly divides the order of the Galois group. This condition limits the ramification indices; this makes all arguments combinatorially rather simple. This condition is also natural for the applications to stable reduction: it is a necessary condition to apply the results of Raynaud ([8]) and Wewers ([9]).

1 The ramification invariant

Let $p > 2$ be a prime number and k an algebraically closed field of characteristic p. We fix an integer $p \leq d < 2p$. Suppose that $f : X \to \mathbb{P}^1_k$ is a non-Galois cover of degree d, defined over k, only branched at $t = \infty$. We call such covers *unramified degree-d covers of the affine line*, for simplicity.

Note that if $f : X \to \mathbb{P}^1_k$ is a Galois cover of degree $p \leq d < 2p$, then $d = p$ and f is an Artin–Schreier cover. In this case everything is known, therefore we exclude this.

Let $g : Y \to \mathbb{P}^1_k$ be the Galois closure of f, and let $G = \mathrm{Gal}(Y, \mathbb{P}^1)$ be its Galois group. It is easy to see that g is also only branched at $t = \infty$. In particular, G is a quasi-p group. The assumption that $p \leq d < 2p$ implies, moreover, that p strictly divides the order of G.

We choose a ramification point $y \in Y$ of g, and let $I = I_y$ be its inertia group. The point y is necessarily wildly ramified. Since p strictly divides the order of G, it follows that I is the extension of a cyclic group C of order prime to p by a cyclic group P of order p.

Let h be the conductor of g and m the order of C. The *ramification invariant* of f (or of g) is defined as $\sigma = h/m$. It is the last jump of the higher ramification groups of y in the upper numbering. The following well-known lemma is proved in [8].

Lemma 1.1 *Let σ be as above, then $\sigma > 1$.*

The goal of this paper is to determine the minimal ramification invariant for certain classes of covers of the affine line. More concretely, we give partial answers to the following questions.

Question 1.2 (a) Let $p \leq d < 2p$. Determine the minimal σ such that there exists a cover $f : X \to \mathbb{P}^1$ only branched at ∞ with ramification invariant σ.

(b) Let G be a quasi-p group. Determine the minimal σ such that there exists a G-Galois cover $f : X \to \mathbb{P}^1$ only branched at ∞ with ramification invariant σ.

We start by studying Question 1.2.(a). We write $\sigma_{d,\min}$ for the minimal ramification invariant occurring for an unramified degree-d cover of the affine line. The following proposition yields a lower bound for $\sigma_{d,\min}$. Before stating it, we need some more notation.

Let $f : X \to \mathbb{P}^1$ be an unramified degree-d cover of the affine line with $p \leq d < 2p$, and let I be the inertia group of a wild ramification point of the Galois closure. Write $|I| = pm$. Choose elements $\theta \in I$ of order m and $\tau \in I$ of order p. Without loss of generality, we may suppose that $\tau = (1\ 2 \cdots p)$. Then θ acting on $\{1, 2, \ldots, p\}$ has one orbit of length 1 and $(p-1)/m_2$ orbits of length m_2. We write $n_1, \ldots, n_t$ for the lengths of the orbits of θ acting on $\{p+1, p+2, \ldots, d\}$. Note that $\sum_{i=1}^t n_i = d - p$. We note that $m = \mathrm{lcm}(m_2, n_1, \ldots, n_t)$.

Proposition 1.3 *(a) The inverse image $f^{-1}(\infty) \subset X$ consists of $t+1$ points, with ramification indices $p, n_1, \ldots, n_t$, respectively.*

(b) The ramification invariant satisfies

$$\sigma = \frac{2g(X) + t + d - 1}{p - 1}.$$

Proof: The ramification indices of $f : X \to \mathbb{P}^1$ are the lengths of the orbits of I acting on $\{1, 2, \ldots, d\}$. The statement of (a) follows therefore immediately from the definition of the n_i and the structure of I.

For (b), we compare the Riemann–Hurwitz formula for $g : Y \to \mathbb{P}^1$ with that for $Y \to X$. Let (h, m) be as above and G the Galois group of g. Then

$$\begin{aligned} 2g(Y) - 2 &= -2|G| + \frac{|G|}{pm}(pm - 1 + h(p-1)) \\ &= (2g(X) - 2)\frac{|G|}{d} + \frac{|G|}{pmd}\left(\sum_{i=1}^{t} n_i(p\frac{m}{n_i} - 1 + h(p-1)\right) + \frac{|G|}{pmd}(p(m-1)). \end{aligned}$$

The statement of (b) follows immediately from this, since $\sigma = h/m$. □

Corollary 1.4 *(a) Suppose $d = p$. Then $\sigma_{d,\min} \geq (d+1)/(p-1)$.*

(b) Suppose that $p + 1 \leq d < 2p$. Then $\sigma_{d,\min} \geq d/(p-1)$.

Proof: Let $d = p$. Let $f : X \to \mathbb{P}^1$ be a degree-d cover of the affine line with ramification invariant σ. Its Galois group is a subgroup of A_p, hence $t = 0$. We conclude that $\sigma = (2g(X) + d - 1)/(p - 1)$. If $g(X) = 0$ then $\sigma = (d-1)/(p-1) = 1$. But this value is not possible by Lemma 1.1. Part (a) follows.

If $p + 1 \leq d < 2p$, we have that $t \geq 1$. As above, we find that $\sigma = (2g(X) + t + d - 1)/(p - 1) \geq d/(p - 1)$. Part (b) follows. □

Remark 1.5 We note that Proposition 1.3 may easily be generalized to covers branched at more than one point. (See for example Lemma 3.2 below.) An other straightforward generalization is to groups G such that p strictly divides the order of the group without the assumption that the degree d of the cover is strictly less than $2p$. In this situation, the permutation representation of the elements τ and θ is different, but the argument extends.

2 Computation of the minimal ramification invariants

In this section, we show that the lower bounds of Corollary 1.4 are indeed sharp for $p+1 \leq d < 2p$. Here we rely on realization results found in the literature. The analogous result for $d = p$ is proved in § 3.

Proposition 2.1 *Let $p+1 \leq d < 2p$. There exists a unique degree-d cover $f : X \to \mathbb{P}^1$ with $\sigma = d/(p+1)$. The Galois group of the Galois closure of f is*

$$\begin{cases} \mathrm{PSL}_2(p) & \text{if } d = p+1, \text{ and } p \geq 5, \\ \mathrm{PSL}_2(8) & \text{if } p = 7 \text{ and } d = p+2, \\ A_d & \text{otherwise.} \end{cases}$$

Proof: We first suppose that $d \geq p+2$. The cover $f : X \to \mathbb{P}^1$ is given by the equation

$$x^d - tx^{d-p} + 1, \qquad (x,t) \mapsto t. \tag{1}$$

This equation has been discovered by Abhyankar ([1]), who also showed the statement on the Galois group.

The computation of the ramification invariant can be found in [6, Theorem 4.9]. In that paper Muskat–Pries also compute the precise structure of the inertia group of a ramification point of the Galois closure of f. Alternatively, one may observe that $g(X) = 0$ and apply Proposition 1.3.(b).

Suppose $p \geq 5$. There exists a $\mathrm{PSL}_2(p)$-Galois cover $g : Y \to \mathbb{P}^1$ with ramification invariant $\sigma = (p+1)/(p-1)$, see for example [3, Cor. 4.2.6]. Let $H < \mathrm{PSL}_2(p)$ be the normalizer of a Sylow p-subgroup of $\mathrm{PSL}_2(p)$. Then H is a subgroup of index $p(p-1)/2$. Therefore $f : X := Y/H \to \mathbb{P}^1$ is a degree-$(p+1)$ cover which is only branched at one point. Alternatively, this may also be seen by considering the equation (1) for $d = p+1$.

We now show the uniqueness of the cover. The assumptions $\deg(f) = d$ and $\sigma = d/(p-1)$ imply that $t = 1$ and $g(X) = 0$, where t is as in Proposition 1.3. In particular, $|f^{-1}(\infty)|$ consist of two points, with ramification indices p and $d-p$, respectively.

Choose a parameter x on X such that $x = \infty$ (resp. $x = 0$) is the ramification point of order p (resp. of order $d-p$). Then f may be given by

$$t = f(x) = \frac{h(x)}{x},$$

where $h(x) := \sum_{i=1}^{p+1} c_i x^i$ is a polynomial of degree $p+1$. The assumption that f is unbranched outside $t = \infty$ implies that $xh' - h = 0$. We conclude that $h(x) = c_0 + c_1 x^{p+1}$ with $c_0 c_1 \not\equiv 0 \pmod p$. After replacing replacing x and t by a multiple, we may assume that $c_0 = c_1 = 1$.

$\square$

The following corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2 *Let $G = A_d$ with $p+2 \leq d < 2p$. We exclude $d = p+2$ in the case that $p = 7$. The minimal ramification invariant of an unramified cover of the affine line with Galois group A_d is $\sigma = d/(p-1)$.*

Proof: Let $G = A_d$ with d as in the statement of the corollary. Then d is the minimal degree an unramified G-Galois cover of the affine line can have. Therefore we conclude from Proposition 1.3.(b) that the ramification invariant σ of such a cover satisfies

$$\sigma \geq \frac{d}{p-1}.$$

By Proposition 2.1 this value is also attained. $\square$

Remark 2.3 Let p and d be as in the statement of Corollary 2.2. Corollary 2.2 immediately yields an expression for the minimal genus of an A_d-Galois cover $g : Y \to \mathbb{P}^1$ which is only branched at $t = \infty$. For the concrete expression, we refer to [6, Theorem 4.9].

Corollary 2.4 *(a) There does not exist an unramified A_{p+1}-cover of the affine line with ramification invariant $\sigma = (p+1)/(p-1)$ and degree $p+1$.*

(b) There does not exist an unramified A_9-cover of the affine line with ramification invariant $\sigma = 9/6 = 3/2$ and degree 9.

Proof: Proposition 2.1 implies that there is a unique unramified cover of the affine line with degree $(p+1)$ and ramification invariant $\sigma = (p+1)/(p-1)$. But this cover has Galois group $\mathrm{PSL}_2(p)$. This proves (a). Part (b) is proved similarly. $\square$

3 Application of stable reduction

In this section, we compute the minimal ramification invariant $\sigma_{p,\min}$ in degree $d = p$, where $p > 3$ is a prime. We exclude the case that $p = 3$, since all degree-3 covers of the affine line in characteristic 3 are Galois.

We use the theory of stable reduction. Rather than giving an overview of the theory, we illustrate the kind of results one may or may not expect to prove. For a quick introduction to stable reduction in the situation we need it here, we refer to [4, § 4]. To understand the statements of this section, it suffices to accept the statement of the description of the so-called primitive tail covers as a black box.

The reason for only considering $d = p$ in this case is that this is much easier that the general case. For a thorough introduction to stable reduction of Belyi maps which is needed to understand the case $p < d < 2p$, we refer the reader to [9].

A *Belyi map* is a finite morphism $f : X \to \mathbb{P}^1_{\bar{\mathbb{Q}}}$ defined over $\bar{\mathbb{Q}}$ which is branched exactly at the three ordered points $0, 1, \infty$. Here X is a smooth projective curve. Belyi maps can be described topologically as *dessins d'enfant* or combinatorially in terms of generating systems.

Definition 3.1 Fix an integer $d > 1$. A *generating system* of degree d is a triple $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ of permutations $\sigma_i \in S_d$ which satisfy

- $\sigma_1\sigma_2\sigma_3 = 1$,
- $G := \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset S_d$ acts transitively on $\{1, 2, \ldots, d\}$.

The *combinatorial type* of $\boldsymbol{\sigma}$ is a tuple $(d; C(\sigma_1), C(\sigma_2), C(\sigma_3))$, where d is the degree and $C(\sigma_i)$ is the conjugacy class of σ_i in S_d.

Two generating systems $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are *equivalent* if there exists a permutation $\tau \in S_d$ such that $\sigma_i' = \tau\sigma_i\tau^{-1}$ for $i = 1, 2, 3$. A general criterion for the existence of generating systems of given combinatorial type can be found in [2].

Let $\boldsymbol{\sigma}$ be a generating system. The conjugacy $C_i := C(\sigma_i)$ of S_d corresponds to a partition $\sum_{i=1}^{r_i} n_i = d$ of d. The *length* $r_i = r(C_i)$ of C_i is the number of cycles of the elements of C_i. Note that we include the 1-cycles. The nonnegative integer $g := (d + 2 - r_1 - r_2 - r_3)/2$ we call the *genus* of the generating system. Note that $g = g(X)$ is the genus of the corresponding Belyi map $X \to \mathbb{P}^1$ and $r_i = |f^{-1}(x_i)|$ is the cardinality of the inverse image of the ith branch point.

Let p be an odd prime number. In this section, we consider Belyi maps $f : X \simeq \mathbb{P}^1 \to \mathbb{P}^1$ of combinatorial type $(p; p, 2^s, C)$, where p is the conjugacy class of a p-cycle, 2^s is the conjugacy class of the product of s disjoint transpositions, and C is any conjugacy class such that $g(X) = 0$. Such generating systems clearly exist.

Since p divides one of the ramification indices, the cover f has bad reduction to characteristic p ([9, Lemma 1.4.(4)]). We refer to [9, § 1.1] for a detailed definition of bad reduction.

With f we may associate two so-called *primitive tail covers* $\bar{f}_2, \bar{f}_3$ ([4, Lemma 4.2]). The primitive tail cover $\bar{f}_i : \bar{X}_i \simeq \mathbb{P}^1 \to \mathbb{P}^1_k$ is a degree-p cover which is totally branched at $t = \infty$, tamely branched at $t = 0$ and unbranched elsewhere. The tame ramification above $t = 0$ is described by the conjugacy class $C(\sigma_i)$, i.e. it is the *same* as the ramification above the ith branch point of the cover f we started with.

Let σ_i be the ramification invariant corresponding to the wild branch point of $\bar{f}_2$.

The following lemma is a straightforward generalization of Proposition 1.3.

Lemma 3.2 *Let $p \leq d < 2p$ and $f : X \to \mathbb{P}^1$ be a degree-d cover in characteristic p which is wildly branched above ∞, tamely branched above 0, and unbranched elsewhere. Let σ be the ramification invariant of ∞ and define t as in § 2. Let $n_1, \ldots, n_r$ with $\sum_{i=1}^r n_i = d$ be the ramification indices above 0. Then*

$$\sigma = \frac{2g(X) + r + t - 1}{p - 1}.$$

Lemma 3.2 implies in our situation that

$$\sigma_i = \frac{r(C_i) - 1}{p - 1}.$$

In particular, $\sigma_2 = (p - s - 1)/(p - 1)$.

Proposition 3.3 *Let $p > 3$ be a prime number. There exists a cover $\tilde{f} : \bar{Z} \to \mathbb{P}^1$ of degree p in characteristic p which is only branched at ∞ with ramification invariant $\sigma = (p + 1)/(p - 1)$.*

In particular, $\sigma_{p,\min} = (p + 1)/(p - 1)$.

Proof: Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^2$. Let $\bar{Z}$ be the normalization of $\bar{X}_2 \times_{\mathbb{P}^1, \varphi} \mathbb{P}^1$. We obtain a commutative square

$$\begin{array}{ccc} \bar{Z} & \longrightarrow & \bar{X}_2 \\ {\scriptstyle \tilde{f}_2}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}_2} \\ \mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^1. \end{array}$$

The natural map $\tilde{f}_2 : \bar{Z} \to \mathbb{P}^1$ of degree p has ramification invariant $2\sigma_2 = 2(p-s-1)/(p-1)$ and is only branched above ∞. This follows from Abhyankar's Lemma (in the version found in [3, Lemma 2.1.2] or [6, Lemma 4.1]). Lemma 3.2 applied to $\tilde{f}_2$ shows that $g(\bar{Z}) = (p - 1)/2 - s$.

The first statement follows by taking $s = (p - 3)/2$. The second statement follows from the first and Corollary 1.4.(b).

□

In [10] one finds an alternative construction of degree-p maps branched at one point between an elliptic curve and the projective line in characteristic p. Zapponi shows that a given elliptic curve E admits such a map if and only if E is supersingular. Moreover, for given E the map is unique. Therefore a degree-p covers of the affine line with ramification invariant $\sigma = (p+1)/(p-1)$ is unique if and only if $p < 12$. This is very different from what we showed in Proposition2.1 for $p + 1 \leq d < 2p$.

We already constructed a $\mathrm{PSL}_2(p)$-Galois cover $g : Y \to \mathbb{P}^1$ of the affine line with ramification invariant $\sigma = (p + 1)/(p - 1)$ in Proposition 2.1. The group $\mathrm{PSL}_2(p)$ admits a subgroup of index p if and only if $p \in \{5, 7, 11\}$ ([5]). For these primes there exists therefore a degree-p cover of the affine line with $\sigma = (p + 1)/(p - 1)$ and Galois group $\mathrm{PSL}_2(p)$. The previous argument implies that it is unique. For large p I expect that there always exists a degree-p cover of the affine line with $\sigma = (p+1)/(p-1)$ whose Galois group is the alternating group A_p.

These results in this note seem to pose more new questions than they solve. Let me just mention a few.

- What is the minimal ramification invariant for $g = A_p$ or $G = A_{p+1}$?
- What happens for other transitive permutation groups which are also quasi-p groups? Natural groups to consider are the noncommutative simple groups, like the Mathieu groups or groups of Lie type.
- What is the minimal ramification invariant for A_9-covers in characteristic 7? (I guess this should be $\sigma = 11/6$.)

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