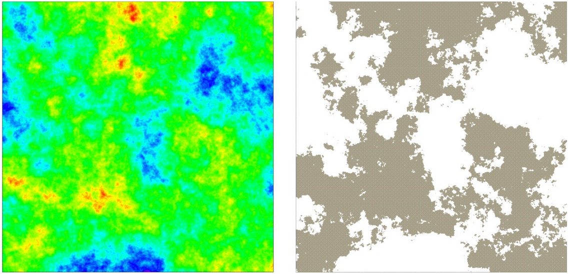

Limit theorems for excursion sets of stationary random fields

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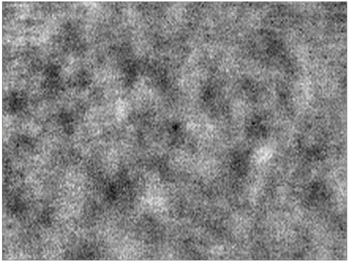
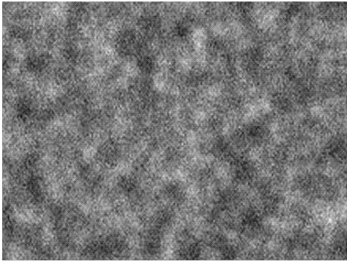
Overview

- Motivation
- Excursion sets of random fields
- Their geometric functionals
- Minkowski functionals of excursion sets: state of art
- CLT for the volume of excursion sets of stationary random fields
  - Second order quasi-associated fields
  - Examples: Shot noise, Gaussian case
  - **PA**- or **NA**-fields (possibly not second order!)
  - Examples: infinitely divisible, max- and α-stable fields
  - Multivariate CLT with a Gaussianity test
- Asymptotics of the mean Minkowski functionals of excursions of non-stationary Gaussian random fields
- Open problems

Motivation



Paper surface
(Voith Paper, Heidenheim)



Simulated Gaussian field

$$\mathrm{E}X(t) = 126$$

$$r(t) = 491 \exp\left(-\frac{\|t\|_2}{56}\right)$$

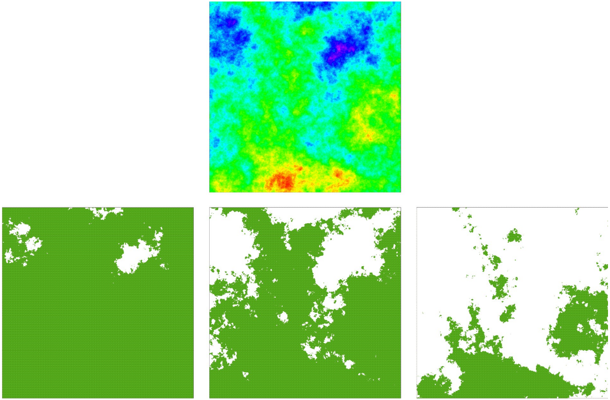
- Is the paper surface Gaussian?

Excursion sets

Let X be a measurable real-valued random field on $\mathbb{R}^d$, $d \geq 1$ and let $W \subset \mathbb{R}^d$ be a measurable subset. Then for $u \in \mathbb{R}$

$$A_u(X, W) := \{t \in W : X(t) \geq u\}$$

is called the excursion set of X in W over the level u.



Centered Gaussian random field on $[0,1]^2$,

$r(t) = \exp(-\|t\|_2/0.3)$,

Levels: $u = -1.0, 0.0, 1.0$

Geometric functionals of excursion sets

Minkowski functionals V_j, $j = 0, \ldots, d$:

- $d = 1$:
  - Length of excursion intervals $V_1\left(A_u\left(X, W\right)\right)$
  - Number of upcrossings $V_0\left(A_u\left(X, W\right)\right)$
- $d \geq 2$:
  - Volume $\left|A_u\left(X, W\right)\right| = V_d\left(A_u\left(X, W\right)\right)$
  - Surface area $\mathcal{H}^{d-1}\left(\partial A_u\left(X, W\right)\right) = 2V_{d-1}\left(A_u\left(X, W\right)\right)$
  - ...
  - Euler characteristic $V_0\left(A_u\left(X, W\right)\right)$, topological measure of "porosity" of $A_u\left(X, W\right)$.
    In $d = 2$:
    $V_0(A) = \#\{\text{connented components of } A\} - \#\{\text{holes of } A\}$

V_j, $j = 0, \ldots, d-2$ are well defined for excursion sets of sufficiently smooth (at least C^2) random fields, see Adler and Taylor (2007).

- Gaussian random fields
  - Moments:
    - Number of upcrossings, $d = 1$: Kac (1943), Rice (1945); Bulinskaya (1961); Cramer & Leadbetter (1967); Belyaev (1972)
    - Minkowski functionals, $d > 1$: Adler (1976, 1981); Wschebor (1983); Adler & Taylor (2007); Azais & Wschebor (2009); S. & Zaporozhets (2012)
  - CLTs:
    - Stationary processes, $d = 1$: Malevich (1969); Cuzick (1976); Piterbarg (1978); Elizarov (1988); Slud (1994); Kratz (2006)
    - Volume, $d \geq 2$: Ivanov & Leonenko (1989)
    - Surface area, $d \geq 2$: Kratz & Leon (2001, 2010)
    - Surface area, $d \geq 2$, FCLT: Meschenmoser & Shashkin (2011-12), Shashkin (2012)
- Non-Gaussian random fields
  - Moments: Adler, Samorodnitsky & Taylor (2010)
  - CLTs: Bulinski, S. & Timmermann (2012); Karcher (2012); Demichev & Schmidt (2012)

Growing sequence of observation windows

A sequence of compact Borel sets $(W_n)_{n\in\mathbb{N}}$ is called a Van Hove sequence (VH) if $W_n \uparrow \mathbb{R}^d$ with

$$\lim_{n\to\infty} |W_n| = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(0)|}{|W_n|} = 0, \quad r > 0.$$

Theorem (CLT for the volume of A_u at a fixed level $u \in \mathbb{R}$)

Let X be a strictly stationary random field satisfying some additional conditions and $u \in \mathbb{R}$ fixed. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has

$$\frac{|A_u(X, W_n)| - \mathrm{P}(X(0) \geq u) \cdot |W_n|}{\sqrt{|W_n|}} \xrightarrow{\text{d}} \mathcal{N}\left(0, \sigma^2(u)\right)$$

as $n \to \infty$. Here

$$\sigma^2(u) = \int_{\mathbb{R}^d} \operatorname{cov}\left(\mathbb{1}\{X(0) \geq u\}, \mathbb{1}\{X(t) \geq u\}\right) dt.$$

Second order quasi-associated random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ have the following properties:

- square-integrable
- has a continuous covariance function
  $r(t) = \mathrm{Cov}(X(o), X(t))$, $t \in \mathbb{R}^d$
- $|r(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ for some $\alpha > 3d$ as $\|t\|_2 \to \infty$
- $X(0)$ has a bounded density
- quasi-associated.

Then $\sigma^2(u) \in (0, \infty)$ (Bulinski, S., Timmermann (2012)).

Quasi-association

A random field $X = \left\{X(t), t \in \mathbb{R}^d\right\}$ with finite second moments is called quasi-associated if

$$\left|\operatorname{cov}\left(f\left(X_I\right), g\left(X_J\right)\right)\right| \leq \sum_{i \in I} \sum_{j \in J} \operatorname{Lip}_i(f) \operatorname{Lip}_j(g) \left|\operatorname{cov}\left(X(i), X(j)\right)\right|$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any Lipschitz functions $f : \mathbb{R}^{\operatorname{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\operatorname{card}(J)} \to \mathbb{R}$ where $X_I = \{X(t), t \in I\}$, $X_J = \{X(t), t \in J\}$.

Idea of the proof of the Theorem: apply a CLT for (BL, θ)-dependent stationary centered square-integrable random fields on $\mathbb{Z}^d$ (Bulinski & Shashkin, 2007).

(BL, θ)-dependence

A real-valued random field $X = \{X(t), t \in \mathbb{R}^d\}$ is called (BL, θ)-dependent, if there exists a sequence $\theta = \{\theta_r\}_{r \in \mathbb{R}_0^+}$, $\theta_r \downarrow 0$ as $r \to \infty$ such that for any finite disjoint sets I, $J \subset T$ with $\operatorname{dist}(I, J) = r \in \mathbb{R}_0^+$, and any functions $f \in BL(|I|)$, $g \in BL(|J|)$, one has

$$|\operatorname{cov}(f(X_I), g(X_J))| \leq \sum_{i \in I} \sum_{j \in J} \operatorname{Lip}_i(f) \operatorname{Lip}_j(g) |\operatorname{cov}(X(i), X(j))| \theta_r,$$

where

$$\theta_r = \sup_{k \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus B_r(k)} |\operatorname{cov}(X(k), X(t))| \, dt.$$

CLT for (BL, θ)-dependent stationary random fields

Theorem (Bulinski & Shashkin, 2007)

Let $Z = \{Z(j), j \in \mathbb{Z}^d\}$ be a (BL, θ)-dependent strictly stationary centered square-integrable random field. Then, for any sequence of regularly growing sets $U_n \subset \mathbb{Z}^d$, one has

$$S(U_n) / \sqrt{|U_n|} \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$$

as $n \to \infty$, with

$$\sigma^2 = \sum_{j \in \mathbb{Z}^d} \operatorname{cov}\left(Z(0), Z(j)\right).$$

Special case - Shot noise random fields

The above CLT holds for a stationary shot noise random field $X = \{X(t), t \in \mathbb{R}^d\}$ given by $X(t) = \sum_{i \in \mathbb{N}} \xi_i \varphi(t - x_i)$ where

- $\{x_i\}$ is a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$
- $\{\xi_i\}$ is a family of i.i.d. non–negative random variables with $\mathrm{E}\,\xi_i^2 < \infty$ and the characteristic function φ_ξ
- $\{\xi_i\}$, $\{x_i\}$ are independent
- $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ is a bounded and uniformly continuous Borel function with

$$\varphi(t) \leq \varphi_0(\|t\|_2) = O\left(\|t\|_2^{-\alpha}\right) \quad \text{as} \quad \|t\|_2 \to \infty$$

for a function $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha > 3d$, and

$$\int\limits_{\mathbb{R}^d} \left| \exp\left\{ \lambda \int_{\mathbb{R}^d} \left(\varphi_\xi(s\varphi(t)) - 1\right) dt \right\} \right| ds < \infty.$$

Special case - Gaussian random fields

Consider a stationary Gaussian random field $X = \{X(t), t \in \mathbb{R}^d\}$ with the following properties:

- $X(0) \sim \mathcal{N}(a, \tau^2)$
- has a continuous covariance function $r(\cdot)$
- $\exists\, \alpha > d : |r(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ as $\|t\|_2 \to \infty$

Special case - Gaussian random fields

Let X be the above Gaussian random field and $u \in \mathbb{R}$. Then,

$$\sigma^2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} \, e^{-\frac{(u-a)^2}{\tau^2(1+r)}} \, dr \, dt,$$

where $\rho(t) = \mathrm{corr}(X(0), X(t))$. In particular, for $u = a$

$$\sigma^2(a) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \arcsin\left(\rho(t)\right) \, dt.$$

Positively or negatively associated random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ have the following properties:

- stochastically continuous (evtl. not second order!)
- $\sigma^2(u) \in (0, \infty)$
- $\mathrm{P}(X(0) = u) = 0$ for the chosen level $u \in \mathbb{R}$
- positively (**PA**) or negatively (**NA**) associated.

Then then above CLT holds (Karcher (2012)).

Association

A random field $X = \left\{X(t), t \in \mathbb{R}^d\right\}$ is called positively (**PA**) or negatively (**NA**) associated if

$$\operatorname{cov}\left(f\left(X_I\right), g\left(X_J\right)\right) \geq 0 \quad (\leq 0, \text{ resp.})$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any bounded coordinatewise non–decreasing functions $f : \mathbb{R}^{\operatorname{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\operatorname{card}(J)} \to \mathbb{R}$ where $X_I = \{X(t), t \in I\}$, $X_J = \{X(t), t \in J\}$.

Special cases

Subclasses of **PA** or **NA**

- infinitely divisible
- max-stable
- α-stable

random fields

Special cases: Max-stable random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary max-stable random field with spectral representation

$$X(t) = \max_{i \in \mathbb{N}} \xi_i f_t(y_i), \quad t \in \mathbb{R}^d,$$

where $f_t : E \to \mathbb{R}_+$ is a measurable function defined on the measurable space (E, μ) for all $t \in \mathbb{R}^d$ with

$$\int_E f_t(y)\, \mu(dy) = 1, \quad t \in \mathbb{R}^d,$$

and $\{(\xi_i, y_i)\}_{i \in \mathbb{N}}$ is the Poisson point process on $(0, \infty) \times E$ with intensity measure $\xi^{-2} d\xi \times \mu(dy)$. Assume that

$$\int_{\mathbb{R}^d} \int_E \min\{f_0(y), f_t(y)\}\, \mu(dy)\, dt < \infty$$

and $\|f_s - f_t\|_{L^1} \to 0$ as $s \to t$.

Special cases: α-stable random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary α-stable random field ($\alpha \in (0,2)$, for simplicity $\alpha \neq 1$) with spectral representation

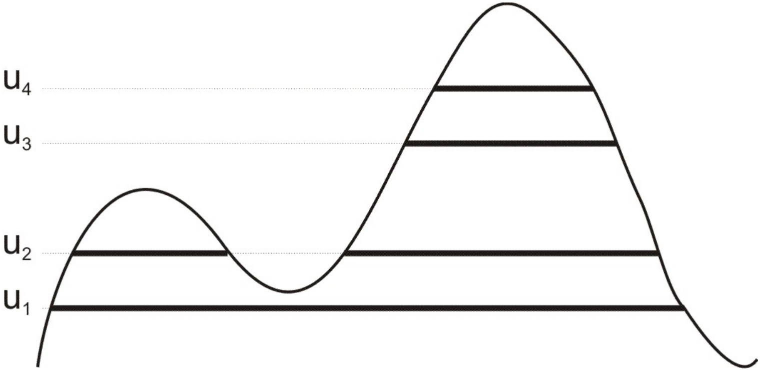
$$X(t) = \int_E f_t(x)\,\Lambda(dx), \quad t \in \mathbb{R}^d,$$

where Λ is a centered independently scattered α–stable random measure on space E with control measure m and skewness intensity $\beta : E \to [-1,1]$, $f_t : E \to \mathbb{R}_+$ is a measurable function on (E, m) for all $t \in \mathbb{R}^d$ with

$$\int_{\mathbb{R}^d} \left(\int_E \min\{|f_0(x)|^\alpha, |f_t(x)|^\alpha\}\, m(dx) \right)^{1/(1+\alpha)} dt < \infty$$

and $\int_E |f_s(x) - f_t(x)|^\alpha\, m(dx) \to 0$ as $s \to t$.

Multi-dimensional CLT



$$S_{\vec{u}}(W_n) = \left(|A_{u_1}(X, W_n)|, \ldots, |A_{u_r}(X, W_n)|\right)^\top$$

$$\Psi(\vec{u}) = \left(\Psi((u_1 - a)/\tau), \ldots, \Psi((u_r - a)/\tau)\right)^\top$$

Theorem (Multi-dimensional CLT)

Let X be the above Gaussian random field and $u_k \in \mathbb{R}$, $k = 1, \dots, r$. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has

$$|W_n|^{-1/2} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u}) \, |W_n| \right) \xrightarrow{d} \mathcal{N}(0, \Sigma(\vec{u}))$$

as $n \to \infty$. Here, $\Sigma(\vec{u}) = (\sigma_{lm}(\vec{u}))_{l,m=1}^{r}$ with

$$\sigma_{lm}(\vec{u}) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} \exp\left\{ -\frac{(u_l-a)^2 - 2r(u_l-a)(u_m-a) + (u_m-a)^2}{2\tau^2(1-r^2)} \right\} dr\, dt.$$

Theorem (Statistical version of the CLT)

Let X be the above Gaussian random field, $u_k \in \mathbb{R}$, $k = 1, \dots, r$ and $(W_n)_{n\in\mathbb{N}}$ be a sequence of VH-growing sets. Let $\hat{C}_n = (\hat{c}_{nlm})_{l,m=1}^{r}$ be statistical estimates for the nondegenerate asymptotic covariance matrix $\Sigma(\vec{u})$, such that for any $l, m = 1, \dots, r$

$$\hat{c}_{nlm} \xrightarrow{p} \sigma_{lm}(\vec{u}) \text{ as } n \to \infty.$$

Then

$$\hat{C}_n^{-1/2} \, |W_n|^{-1/2} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u}) \, |W_n|\right) \xrightarrow{d} \mathcal{N}(0, I).$$

Hypothesis testing

$$H_0 : X \text{ Gaussian vs. } H_1 : X \text{ Non-Gaussian}$$

Test statistic:

$$T = |W_n|^{-1} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u})\, |W_n|\right)^\top \hat{C}_n^{-1} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u})\, |W_n|\right)$$

We know $T \xrightarrow{\mathrm{d}} \chi_r^2$. Reject null-hypothesis if $T > \chi^2_{r,1-\nu}$.

Numerical results

Series	FTR6.3	FTR6.6	Sim. Gaussian
Resolution	218*x*138	218*x*138	218*x*138
Realizations	100	100	100
1 level			
Rejected fields ($\nu = 1\%$)	0	0	1
3 levels			
Rejected fields ($\nu = 1\%$)	5	9	3
5 levels			
Rejected fields ($\nu = 1\%$)	20	21	3
7 levels			
Rejected fields ($\nu = 1\%$)	34	31	5
9 levels			
Rejected fields ($\nu = 1\%$)	62	60	5

- FCLT (variable $u \in \mathbb{R}$):
  - Volume for second order **A**-random fields with a.s. continuous paths and bounded density in Skorokhod space: Meschenmoser & Shashkin (2011)
  - Volume for random fields with a.s. continuous paths and bounded density in Skorokhod space (evtl. not second order!): Karcher (2012)
  - Surface area for isotropic Gaussian random fields in $L^2(\mathbb{R})$: Meschenmoser & Shashkin (2012)
  - Surface area for isotropic C^1-smooth Gaussian random fields in $C(\mathbb{R})$: Shashkin (2012)
- CLT for the volume as level $u \to \infty$
  - Isotropic Gaussian random fields: Ivanov & Leonenko (1989)
  - **PA**-random fields: Demichev & Schmidt (2012)

Non-stationary Gaussian random fields

Let $X = \{X(t),\ t \in W\}$ be a centered smooth Gaussian random field with variance $\sigma^2(t)$ where $W = \prod_{j=1}^{d}[0, a_j]$, $a_1, \dots, a_d > 0$. Assume that σ has a unique global maximum at the origin and $\sigma_i'(0) < 0$ for $i = 1, \dots, d$. Let $A_u = A_u(X; W)$.

Problem: find the asymptotic of $\mathrm{E}\,V_j(A_u)$ as $u \to +\infty$ (S., Zaporozhets (2012))

Non-stationary Gaussian random fields

Does it hold the Euler-Poincaré heuristic

$$\left| \mathrm{P}\left(\sup_{t\in W} X(t) > u\right) - \mathrm{E}\, V_0\,(A_u)\right| \leq c_0 \exp\{-u^2(1+\alpha)/2\}$$

for some $c_0, \alpha > 0$ as in the case of Gaussian fields that are

- stationary (for any u) (Adler (1981))
- non-stationary with $\sigma(t) \equiv \sigma$ for $u \to \infty$ (Adler, Taylor (2007))
- non-stationary with $\sigma(t)$ having a unique point of maximum in *int* W for $u \to \infty$ (Azais, Wschebor (2009))?

Notation

For $X \in \mathcal{C}^2(W)$ a.s., put

X_i', σ_i'	partial derivatives of X, σ with respect to ith variable
X_{ij}''	$\partial^2 X / \partial t_i \partial t_j$
X''	$\left(X_{ij}''\right)_{1 \leqslant i,j \leqslant d}$, the Hessian matrix of X
∇X	$(X_1', \ldots, X_d')^\top$
Z	vector $(X, \nabla X)^\top$.
$\Sigma(t)$	covariance matrix of $\{Z(t),\ t \in W\}$
Φ	c.d.f. of $\mathcal{N}(0,1)$
Ψ	$1 - \Phi$

Supremum probability

Lemma

Suppose that $X \in \mathcal{C}^1(W)$ a.s. Then

$$\mathrm{P}(\sup_{t \in W} X(t) > u) = \Psi\left(\frac{u}{\sigma(0)}\right) \cdot [1 + o(1)], \quad u \to +\infty.$$

Idea of the proof: It follows from the known result of Talagrand (1988).

Asymptotic of the mean Euler number

Theorem

Suppose that

- $X \in \mathcal{C}^2(W)$ *a.s.,*
- $Z(t), (X_i'(t), X_{ij}''(t))$ *have nondegenerate distributions for all* $t \in W$
- $$\mathrm{P}\left(\exists\, u > 0, t \in W : X(t) = u, \nabla X(t) = 0, \det X''(t) = 0\right) = 0.$$

Then

$$\mathrm{E}\, V_0(A_u) = \Psi\left(\frac{u}{\sigma(0)}\right) \cdot [1 + O(u^{-1})], \quad u \to +\infty.$$

Idea of the proof: Use the Laplace method together with Morse theorem.

Asymptotic of the mean volume

Theorem

Suppose that

- $\sigma \in \mathcal{C}(W)$,
- $\sigma \in \mathcal{C}^2$ *in some neighborhood of the origin.*

Then

$$\mathsf{E} V_d(A_u) = \frac{C}{u^{2d}} \Psi\left(\frac{u}{\sigma(0)}\right) \cdot [1 + o(1)], \quad u \to +\infty,$$

where

$$C = \frac{(-1)^d}{(2\pi)^{(d-1)/2}} \frac{\sigma^{3d+2}(0)}{\prod_{j=1}^d \sigma_j'(0)}.$$

Asymptotic of the mean surface area

Theorem

Suppose that

- $X \in \mathcal{C}^1(W)$ *a.s.,*
- $\sigma \in \mathcal{C}^2$ *in some neighborhood of the origin,*
- $\sigma(t) > 0$ *for all* $t \in W$.

Then

$$\mathrm{E}\, V_{d-1}(A_u) = \frac{C}{u^{2d-1}} \Psi\left(\frac{u}{\sigma(0)}\right) \cdot [1 + o(1)], \quad u \to +\infty,$$

where $C = \frac{(-1)^d}{2} \mathrm{E} \left\| \nabla \frac{X}{\sigma}(0) \right\| \frac{\sigma^{3d+1}(0)}{\prod_{j=1}^d \sigma_j'(0)}$.

Idea of the proof: Use the Laplace method and the formula for $\mathrm{E}\, V_{d-1}(A_u)$ (Ibragimov & Zaporozhets, 2010).

Open problems

- CLT for other Minkowski functionals of stationary Gaussian random fields (e.g., for the Euler number!)
- Asymptotic for $\mathrm{E}\, V_j(A_u)$, $j = 1, \ldots, d-2$ as $u \to \infty$ for non-stationary Gaussian random fields
- Rate of convergence in the Euler-Poincaré heuristic
- More general observation windows W and non-stationary Gaussian fields on stratified manifolds

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Thank you for your attention!