Kernel function estimation for stable moving average random fields

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Abstract

We propose a method to estimate the symmetric kernel function with compact support of an α-stable moving average random field with $\alpha \in (1, 2]$. We prove consistency of the estimator and apply the method to a numerical example.

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1. Introduction

Let $1 < \alpha \leq 2$ and $X = \{X(t), t \in \mathbb{R}^d\}$ be a real-valued α-stable moving average random field of the form

$$X(t) = \int_{\mathbb{R}^d} f(t-x)M(dx), \quad t \in \mathbb{R}^d, \tag{1}$$

where $f \in L^\alpha(\mathbb{R}^d)$ and M is an α-stable random measure with Lebesgue control measure, see Samorodnitsky and Taqqu (1994, Chapter 3). A minimum contrast method to estimate the kernel function f parametrically is provided by Karcher et al. (2009). In this paper, we are interested in estimating f non-parametrically.

Assume that f is symmetric, i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}^d$. If $\alpha = 2$, then X is a Gaussian random field with covariance function $C : \mathbb{R}^d \to \mathbb{R}$ given by

$$C(h) = \mathrm{Cov}(X(h), X(0)) = 2 \int_{\mathbb{R}^d} f(h-x) f(x) dx,$$

cf. Samorodnitsky and Taqqu (1994, Example 2.7.2 and Proposition 3.5.2). By applying the Fourier transform $\mathcal{F}$, we obtain $\mathcal{F}(C) = 2 \cdot \mathcal{F}(f)^2$ and therefore

$$f(x) = \pm 2^{-1/2} \mathcal{F}^{-1}\left(\sqrt{\mathcal{F}(C)}\right)(x), \quad x \in \mathbb{R}^d. \tag{2}$$

The last formula can be used to estimate f based on an estimate of the covariance function C. However if $\alpha < 2$, X does not have a second moment such that (2) cannot be applied any more. In this case, we can replace C by the so-called *covariation function* $\kappa : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\kappa(h) := \int_{\mathbb{R}^d} f(h-x) f(x)^{\langle \alpha - 1 \rangle} dx, \tag{3}$$

where $a^{\langle p \rangle} = |a| \mathrm{sign}(a)$ for real numbers a and p. This is the most common generalization of the covariance function, see Samorodnitsky and Taqqu (1994, Chapter 2.7). Notice that $\kappa(h) = 1/2 \cdot C(h)$ for all $h \in \mathbb{R}^d$ if $\alpha = 2$, cf. Samorodnitsky and Taqqu (1994, Proposition 3.5.2). Unfortunately, the Fourier transform of the right hand side in (3) does not factorize any more. Therefore, we propose a recursive approach in the next section in order to retrieve f based on (3). In the following, we assume that f is a step function which is compactly supported on a cube K. For the analysis of the approximation error when using step functions to approximate the kernel function, see Karcher et al. (2011a).

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2. Non-parametric estimation of the kernel function

Suppose that the symmetric kernel function f has the form

$$f(x) = \sum_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} f_{\mathbf{i}} \mathbb{1}_{\Delta_{\mathbf{i}}}(x), \quad x \in \mathbb{R}^d,$$

where $\mathbf{1} = (1, \ldots, 1)^\intercal \in \mathbb{R}^d$, $\mathbf{i} = (i_1, \ldots, i_d)^\intercal \in \mathbb{N}^d$, $\mathbf{n} = (n, \ldots, n)^\intercal \in \mathbb{N}^d$, $f_{\mathbf{i}} \in \mathbb{R}$, $\Delta_{\mathbf{i}} \subset \mathbb{R}^d$ are pairwise disjoint bounded Borel sets such that $\cup_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}}\Delta_{\mathbf{i}} = K$ and inequalities are understood to be componentwise. The covariation function of the field (1) is given by

$$\kappa(h) = \sum_{\mathbf{1}\le\mathbf{i},\mathbf{j}\le\mathbf{n}} f_{\mathbf{i}} f_{\mathbf{j}}^{\langle\alpha-1\rangle} \nu_d(\Delta_{\mathbf{i}} \cap (\Delta_{\mathbf{j}} + h)), \tag{4}$$

where ν_d denotes the d-dimensional Lebesgue measure. If we replace κ in (4) by some estimate $\hat{\kappa}$, we obtain for fixed locations $h_1, \ldots, h_p \in \mathbb{R}^d$, $p \in \mathbb{N}$, a non-linear system of equations

$$\hat{\kappa}(h_q) = \sum_{\mathbf{1}\le\mathbf{i},\mathbf{j}\le\mathbf{n}} f_{\mathbf{i}} f_{\mathbf{j}}^{\langle\alpha-1\rangle} \nu_d(\Delta_{\mathbf{i}} \cap (\Delta_{\mathbf{j}} + h_q)), \quad q = 1, \ldots, p, \tag{5}$$

for the unknown values $f_{\mathbf{1}}, \ldots, f_{\mathbf{n}}$. The system of equations (5) can be solved numerically if a solution exists, e.g. with the Levenberg-Marquardt algorithm (see Moré (1978)) or with the recursive algorithm we present in the next section. By applying the recursive algorithm to (4), it immediately follows that for an appropriate choice of $h_1, \ldots, h_p$, (4) has exactly two solutions. However, the existence of a solution of (5) cannot be guaranteed because estimators are involved in the left hand side. We therefore include a cut-off technique in the recursive algorithm which remedies this problem and delivers two (approximate) solutions. The choice of the eligible solution is discussed after the presentation of the algorithm.

2.1. Recursive algorithm

We restrict to $d = 2$. The extension to $d \in \mathbb{N}$ is straightforward. Let us assume that the sets $\Delta_{\mathbf{i}}$, $\mathbf{1} \le \mathbf{i} \le \mathbf{n}$, are squares with equal side lengths. Then we can represent the kernel function by the values $f_{(1,1)}, \ldots, f_{(n,n)}$, $n \in \mathbb{N}$, as shown in Figure 1.

$f_{(1,1)}$	$f_{(1,2)}$	$\cdots$	$f_{(1,n-1)}$	$f_{(1,n)}$
$f_{(2,1)}$	$f_{(2,2)}$	$\cdots$	$f_{(2,n-1)}$	$f_{(2,n)}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$f_{(2,n)}$	$f_{(2,n-1)}$	$\cdots$	$f_{(2,2)}$	$f_{(2,1)}$
$f_{(1,n)}$	$f_{(1,n-1)}$	$\cdots$	$f_{(1,2)}$	$f_{(1,1)}$

Figure 1: The shape of the kernel function f for $d = 2$.

Under the preceding assumptions, the system of non-linear equations (5) can be solved with the following algorithm which is based on two steps. In the first step, the first and last line of the kernel matrix $\{f_{(i,j)}\}_{i,j=1}^n$ is estimated. In the second step, the remaining entries are calculated recursively.

Let $\{\varepsilon_l\}_{l\in\mathbb{N}}$ be a sequence such that

$$\varepsilon_l \to 0, \quad l \to \infty,$$

where l is the size of the sample $(X(t_1), \ldots, X(t_l))$ with which the covariation function κ is estimated.

Step 1. There exists an $h_{(1,1)} \in \mathbb{R}^2$ such that $\kappa(h_{(1,1)}) = |f_{(1,1)}|^{\alpha}$, cf. Figure 2. Therefore, we set

$$\hat{f}_{(1,1)} = \pm|\hat{\kappa}(h_{(1,1)})|^{1/\alpha} \cdot \mathbb{1}_{|\hat{\kappa}(h_{(1,1)})| \geq \varepsilon_l}. \tag{6}$$

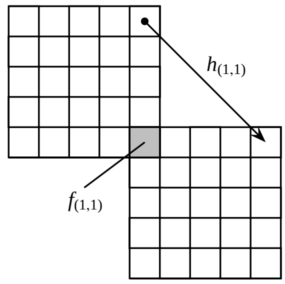

Figure 2: Estimation of $f_{(1,1)}$.

Assume first that $\hat{f}_{(1,p)} = 0$ for $1 \leq p < \lceil n/2 \rceil$. Here, $\lceil\cdot\rceil$ denotes the ceil function. Then there exists an $h_{(1,p+1)} \in \mathbb{R}^2$ such that

$$\kappa(h_{(1,p+1)}) = f_{(1,1)} f_{(1,2p+1)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,p)} f_{(1,p+2)}^{\langle\alpha-1\rangle} + |f_{(1,p+1)}|^{\alpha} + f_{(1,p+2)} f_{(1,p)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,2p+1)} f_{(1,1)}^{\langle\alpha-1\rangle},$$

cf. Figure 3. Therefore, we set

$$\begin{aligned} \hat{f}_{(1,p+1)} &= \pm\left| |\hat{\kappa}(h_{(1,p+1)})| \cdot \mathbb{1}_{|\hat{\kappa}(h_{(1,p+1)})| \geq \varepsilon_l} - \sum_{i=1}^{p} \hat{f}_{(1,i)} f_{(1,2p+2-i)}^{\langle\alpha-1\rangle} - \sum_{i=p+2}^{2p+1} f_{(1,i)} \hat{f}_{(1,2p+2-i)}^{\langle\alpha-1\rangle} \right|^{1/\alpha} \\ &= \pm|\hat{\kappa}(h_{(1,p+1)})|^{1/\alpha} \cdot \mathbb{1}_{|\hat{\kappa}(h_{(1,p+1)})| \geq \varepsilon_l}. \end{aligned} \tag{7}$$

If $\hat{f}_{(1,p)} = 0$ for $\lceil n/2 \rceil \leq p < n$, then there exists an $h_{(1,p+1)} \in \mathbb{R}^2$ such that

$$\kappa(h_{(1,p+1)}) = f_{(1,2p+2-n)} f_{(1,n)}^{\langle\alpha-1\rangle} + \ldots + |f_{(1,p+1)}|^{\alpha} + \ldots + f_{(1,n)} f_{(1,2p+2-n)}^{\langle\alpha-1\rangle}$$

and we set

$$\begin{aligned} \hat{f}_{(1,p+1)} &= \pm\left| |\hat{\kappa}(h_{(1,p+1)})| \cdot \mathbb{1}_{|\hat{\kappa}(h_{(1,p+1)})| \geq \varepsilon_l} - \sum_{i=2p+2-n}^{p} \hat{f}_{(1,i)} f_{(1,2p+2-i)}^{\langle\alpha-1\rangle} - \sum_{i=p+2}^{n} f_{(1,i)} \hat{f}_{(1,2p+2-i)}^{\langle\alpha-1\rangle} \right|^{1/\alpha} \\ &= \pm|\hat{\kappa}(h_{(1,p+1)})|^{1/\alpha} \cdot \mathbb{1}_{|\hat{\kappa}(h_{(1,p+1)})| \geq \varepsilon_l}. \end{aligned} \tag{8}$$

We refer to Remark 2 for a method to decide whether to take the positive or negative value in (6), (7) and (8).

Assume now that for some $q \in \{1, \ldots, n\}$, it holds $\hat{f}_{(1,q)} \neq 0$ and $\hat{f}_{(1,j)} = 0$ for $j = 1, \ldots, q-1$. Suppose that we have estimated $f_{(1,q)}, \ldots, f_{(1,r)}$ for some $r \in \{q, \ldots, n-1\}$. By successively moving the upper square in Figure 3 to the right, we find some $h_{(1,r+1)} \in \mathbb{R}^2$ such that

$$\kappa(h_{(1,r+1)}) = \begin{cases} f_{(1,1)} f_{(1,q+r)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,q)} f_{(1,r+1)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,r+1)} f_{(1,q)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,q+r)} f_{(1,1)}^{\langle\alpha-1\rangle}, & r \leq n-q, \\ f_{(1,q-n+r+1)} f_{(1,n)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,q)} f_{(1,r+1)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,r+1)} f_{(1,q)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,n)} f_{(1,q-n+r+1)}^{\langle\alpha-1\rangle}, & r > n-q. \end{cases} \tag{9}$$

Plugging the estimates $\hat{\kappa}(h_{(1,r-1)}), \hat{f}_{(1,1)}, \ldots, \hat{f}_{(1,r)}$ in (9) and solving for $f_{(1,r+1)}$ yields with $\hat{a}(r) = \sum_{i=q+1}^{r} \hat{f}_{(1,i)} \hat{f}_{(1,r+q+1-i)}^{\langle\alpha-1\rangle}$ the estimation equation

$$\hat{f}_{(1,q)} f_{(1,r+1)}^{\langle\alpha-1\rangle} + f_{(1,r+1)} \hat{f}_{(1,q)}^{\langle\alpha-1\rangle} = \hat{\kappa}(h_{(1,r+1)}) - \hat{a}(r) \tag{10}$$

because $\hat{f}_{(1,1)} = \ldots = \hat{f}_{(1,q-1)} = 0$. The last equation which is non-linear with respect to $f_{(1,r+1)}$ can be solved with the Newton-Raphson algorithm, for example (see e.g. Kelly (2003)).

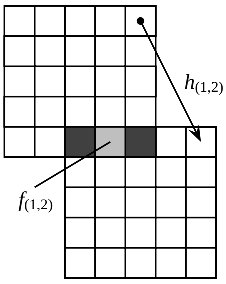

Figure 3: Estimation of $f_{(1,2)}$ when $\hat{f}_{(1,1)} = 0$. The estimate is obtained from the equation $\hat{\kappa}(h_{(1,2)}) = \hat{f}_{(1,1)} f_{(1,3)}^{\langle\alpha-1\rangle} + |f_{(1,2)}|^{\alpha} + f_{(1,3)} \hat{f}_{(1,1)}^{\langle\alpha-1\rangle} = |f_{(1,2)}|^{\alpha}$.

Remark 1. When using the Newton-Raphson algorithm to solve (10), an additional approximation error occurs which propagates via $\hat{a}(r)$. However, the error in each step can be made arbitrarily small and the problem of finding the solution of (10) is stable with respect to small changes in $\hat{a}(r)$. Namely the right hand side of (10) is a continuous function in $f_{(1,r+1)}$ which is strictly monotonic increasing or strictly monotonic decreasing.

Step 2. Assume that we have estimated $f_{(i,j)}$ for $1 \leq i \leq s-1$ and $1 \leq j \leq n$, where $2 \leq s < \lceil n/2 \rceil$. We choose $h_{(s,1)} \in \mathbb{R}^2$ such that the estimated value $\hat{f}_{(1,q)} \neq 0$ overlaps with the entry $f_{(s,1)}$, cf. Figure 4.

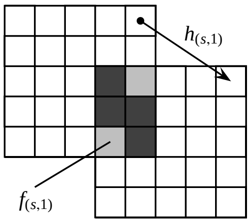

Figure 4: Estimation of $f_{(3,1)}$. Assume that $q = 2$, i. e. $\hat{f}_{(1,1)} = 0$ and $\hat{f}_{(1,2)} \neq 0$ and the first and second row of the kernel matrix have already been estimated. The lightgray squares represent the values $f_{(3,1)}$ and $f_{(1,2)}$. The equation to estimate $f_{(3,1)}$ is
$\hat{\kappa}(h_{(3,1)}) = \hat{f}_{(1,1)} f_{(3,2)}^{\langle\alpha-1\rangle} + \hat{f}_{(1,2)} f_{(3,1)}^{\langle\alpha-1\rangle} + \hat{f}_{(2,1)} \hat{f}_{(2,2)}^{\langle\alpha-1\rangle} + \hat{f}_{(2,2)} \hat{f}_{(2,1)}^{\langle\alpha-1\rangle} + f_{(3,1)} \hat{f}_{(1,2)}^{\langle\alpha-1\rangle} + f_{(3,2)} \hat{f}_{(1,1)}^{\langle\alpha-1\rangle} = \hat{f}_{(1,2)} f_{(3,1)}^{\langle\alpha-1\rangle} + \hat{f}_{(2,1)} \hat{f}_{(2,2)}^{\langle\alpha-1\rangle} + \hat{f}_{(2,2)} \hat{f}_{(2,1)}^{\langle\alpha-1\rangle} + f_{(3,1)} \hat{f}_{(1,2)}^{\langle\alpha-1\rangle}$.

It holds

$$\kappa(h_{(s,1)}) = f_{(1,1)} f_{(s,q)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,q)} f_{(s,1)}^{\langle\alpha-1\rangle} + \sum_{i=2}^{s-1} \sum_{j=1}^{q} f_{(i,j)} f_{(s+1-i,q+1-j)}^{\langle\alpha-1\rangle} + f_{(s,1)} f_{(1,q)}^{\langle\alpha-1\rangle} + \ldots + f_{(s,q)} f_{(1,1)}^{\langle\alpha-1\rangle}.$$

From the last equation, we obtain the estimation equation

$$f_{(s,1)} \hat{f}_{(1,q)}^{\langle\alpha-1\rangle} + \hat{f}_{(1,q)} f_{(s,1)}^{\langle\alpha-1\rangle} = \hat{\kappa}(h_{(s,1)}) - \sum_{i=2}^{s-1} \sum_{j=1}^{q} \hat{f}_{(i,j)} \hat{f}_{(s+1-i,q+1-j)}^{\langle\alpha-1\rangle} \tag{11}$$

since $\hat{f}_{(1,1)}, \ldots, \hat{f}_{(1,q-1)} = 0$.

Assume now that we have estimated $f_{(i,j)}$ for $(i, j) \in \{(k, l) : 1 \leq k \leq s-1, 1 \leq l \leq n\} \cup \{(s, l) : 1 \leq l \leq r\}$, where $2 \leq s < \lceil n/2 \rceil$ and $1 \leq r \leq n-1$. By successively moving the upper square in Figure 4 to the right, we find some $h_{(s,r+1)} \in \mathbb{R}^2$ such that similarly to (9)

$$\kappa(h_{(s,r+1)}) = \begin{cases} f_{(s,1)} f_{(1,q+r)}^{\langle\alpha-1\rangle} + \ldots + f_{(s,q+r)} f_{(1,1)}^{\langle\alpha-1\rangle} + \sum\limits_{i=2}^{s-1} \sum\limits_{j=1}^{q+r} f_{(i,j)} f_{(s+1-i,q+r+1-j)}^{\langle\alpha-1\rangle} & \\ \qquad + f_{(1,q+r)} f_{(s,1)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,1)} f_{(s,q+r)}^{\langle\alpha-1\rangle}, & r \leq n-q, \\ f_{(s,q-n+r+1)} f_{(1,n)}^{\langle\alpha-1\rangle} + \ldots + f_{(s,n)} f_{(1,q-n+r+1)}^{\langle\alpha-1\rangle} + \sum\limits_{i=2}^{s-1} \sum\limits_{j=q-n+r+1}^{n} f_{(i,j)} f_{(s+1-i,q+r+1-j)}^{\langle\alpha-1\rangle} & \\ \qquad + f_{(1,n)} f_{(s,q-n+r+1)}^{\langle\alpha-1\rangle} + \ldots + f_{(1,q-n+r+1)} f_{(s,n)}^{\langle\alpha-1\rangle}, & r > n-q. \end{cases}$$

The corresponding estimation equation is

$$
\begin{aligned}
f_{(s,r+1)}\hat{f}_{(1,q)}^{\langle\alpha-1\rangle} + \hat{f}_{(1,q)}f_{(s,r+1)}^{\langle\alpha-1\rangle} &= \hat{\kappa}(h_{(s,r+1)}) - \sum_{i=2}^{s-1}\sum_{j=\max\{1,q-n+r+1\}}^{\min\{q+r,n\}} \hat{f}_{(i,j)}\hat{f}_{(s+1-i,q+r+1-j)}^{\langle\alpha-1\rangle} - \sum_{j=q+1}^{r}\hat{f}_{(s,r+q+1-j)}\hat{f}_{(1,j)}^{\langle\alpha-1\rangle} \\
&\quad - \sum_{j=q+1}^{r}\hat{f}_{(1,r+q+1-j)}\hat{f}_{(s,j)}^{\langle\alpha-1\rangle}.
\end{aligned}
\tag{12}
$$

2.2. Consistency of the estimator

Theorem 1. *Let $h_{(1,\dots,1)},\dots,h_{(\lceil n/2\rceil,n,\dots,n)}$ be vectors in $\mathbb{R}^d$ as described in Step 1 and Step 2 (for the case $d=2$). Let $\hat{\kappa}(h_{(1,\dots,1)}),\dots,\hat{\kappa}(h_{(\lceil n/2\rceil,n,\dots,n)})$ be strongly consistent estimators for $\kappa(h_{(1,\dots,1)}),\dots,\kappa(h_{(\lceil n/2\rceil,n,\dots,n)})$ with rate of convergence $O(g(l))$ for some function $g:\mathbb{N}\to\mathbb{R}_+$, where $l\in\mathbb{N}$ is the sample size. Assume that $\varepsilon_l\to 0$ and $g(l)/\varepsilon_l\to 0$ as $l\to\infty$. Then the estimators based on (6)–(8) and (10)–(12) for $f_{(i_1,\dots,i_n)}$ are strongly consistent for $i_1=1,\dots,\lceil n/2\rceil$ and $i_j=1,\dots,n$, $j\in\{2,\dots,n\}$.*

Proof. Without loss of generality, let $d=2$. If $f_{(1,1)}=\kappa(h_{(1,1)})=0$, then

$$
\hat{f}_{(1,1)} = \pm|\hat{\kappa}(h_{(1,1)})|^{1/\alpha}\mathbb{1}_{|\hat{\kappa}(h_{(1,1)})|\geq\varepsilon_l} = \pm|\hat{\kappa}(h_{(1,1)})|^{1/\alpha}\mathbb{1}_{\frac{g(l)}{\varepsilon_l}\frac{|\hat{\kappa}(h_{(1,1)})|}{g(l)}\geq 1} \overset{a.s.}{\to} \pm|\kappa(h_{(1,1)})|^{1/\alpha}\cdot 0 = 0, \quad l\to\infty.
$$

If $\kappa(h_{(1,1)})\neq 0$, then $\hat{f}_{(1,1)} = \pm|\hat{\kappa}(h_{(1,1)})|^{1/\alpha}\mathbb{1}_{|\hat{\kappa}(h_{(1,1)})|\geq\varepsilon_l}\to\pm|\kappa(h_{(1,1)})|^{1/\alpha}=f_{(1,1)}$ almost surely because $\varepsilon_l\to 0$ as $l\to\infty$. Therefore, the estimators (6)–(8) are strongly consistent if the sign of $\hat{f}_{(1,1)}$ is chosen as in Remark 2.

Consider now the estimation equation (10) for $f_{(1,r+1)}$. The estimator $\hat{f}_{(1,r+1)}$ is the root of the function

$$
g(x) = \hat{f}_{(1,q)}x^{\langle\alpha-1\rangle} + \hat{f}_{(1,q)}^{\langle\alpha-1\rangle}x + \hat{a}(r) - \hat{\kappa}(h_{(1,r+1)}).
$$

Notice that g is continuous and either strictly monotonic increasing or strictly monotonic decreasing. Therefore it has exactly one root. Since the coefficients of g are strongly consistent, its root converges almost surely to $f_{(1,r+1)}$, so the estimator based on (10) is strongly consistent. With the same argument we obtain the strong consistency of the estimators based on (11) and (12). □

Remark 2. (a) Let $1<\alpha<2$ and suppose that $(X,Y)^\mathsf{T}$ is an α-stable random vector with spectral measure Γ such that $X\sim S_\alpha(\sigma_X,\beta_X,0)$ and $Y\sim S_\alpha(\sigma_Y,\beta_Y,0)$. For $1\leq p<\alpha$, it holds

$$
\frac{\mathbb{E}\left(XY^{<p-1>}\right)}{\mathbb{E}|Y|^p} = \frac{[X,Y]_\alpha(1-c\cdot\beta_Y)+c\cdot(X,Y)_\alpha}{\sigma_Y^\alpha}, \tag{13}
$$

where $(X,Y)_\alpha = \int_{\mathbb{S}^1} s_1|s_2|^{\alpha-1}\Gamma(ds)$, $\mathbb{S}^1$ denotes the unit circle and

$$
c = c_{\alpha,p}(\beta_Y) = \frac{\tan(\alpha\pi/2)}{1+\beta_Y^2\tan^2(\alpha\pi/2)}\left[\beta_Y\tan(\alpha\pi/2) - \tan\left(\frac{p}{\alpha}\arctan(\beta_Y\tan(\alpha\pi/2))\right)\right],
$$

see Karcher et al. (2011b). If Y is symmetric, i.e. $\beta_Y=0$, then $c=0$. Equation (13) can be used to estimate $\kappa(h)$ as follows.

If X is a symmetric α-stable random field, then

$$
\kappa(h) = \frac{(\mathbb{E}|X(0)|^p)^{\alpha/p-1}\,\mathbb{E}\left(X(0)X(h)^{\langle p-1\rangle}\right)}{c_\alpha(p)^\alpha}, \quad 1<p<\alpha, \tag{14}
$$

where

$$
c_\alpha(p)^p = \frac{2^{p-1}\Gamma(1-p/\alpha)}{p\int_0^\infty u^{-p-1}\sin^2 u\,du},
$$

see Samorodnitsky and Taqqu (1994, Property 1.2.17).

Assume that we can observe the random field on the grid $\mathbb{N}^d$. For $\mathbf{m} = (m_1, \ldots, m_d)^\top \in \mathbb{N}^d$ and $h \in \mathbb{Z}^d$, let $D(\mathbf{m}) = [\mathbf{1}, \mathbf{m}] \cap \mathbb{N}^d$ and $D_h(\mathbf{m}) = D(\mathbf{m}) \cap (D(\mathbf{m}) - h)$. We consider the estimators

$$\widehat{\mathbb{E}|X(0)|^p} = \frac{1}{|\mathbf{m}|} \sum_{t \in D(\mathbf{m})} |X(t)|^p, \tag{15}$$

$$\mathbb{E}\,(\widehat{X(0)X(h)^{\langle p-1\rangle}}) = \frac{1}{\operatorname{card}(D_h(\mathbf{m}))} \sum_{t \in D_h(\mathbf{m})} X(t)X(t+h)^{\langle p-1\rangle}, \tag{16}$$

where $|\mathbf{m}| = m_1 \cdot \ldots \cdot m_d$ and $\operatorname{card}(D_h(\mathbf{m}))$ denotes the cardinality of $D_h(\mathbf{m})$. Since the support of the kernel function of X is bounded, the random fields $\{X(t), t \in \mathbb{R}^d\}$ and $\{X(t)X(t+h), t \in \mathbb{R}^d\}$ are $\mathcal{M}$-dependent, cf. Samorodnitsky and Taqqu (1994, Theorem 3.5.3). If $p < \beta < \alpha$, then $\mathbb{E}\,(|X(0)|^p)^{\beta/p} = \mathbb{E}|X(0)|^\beta < \infty$ and

$$\lim_{\mathbf{m}\to\infty} |\mathbf{m}|^{1-\gamma} \left(\widehat{\mathbb{E}|X(0)|^p} - \mathbb{E}|X(0)|^p\right) = 0 \quad a.s.$$

$$\lim_{\mathbf{m}\to\infty} \operatorname{card}(D_h(\mathbf{m}))^{1-\gamma} \left(\mathbb{E}\,(\widehat{X(0)X(h)^{\langle p-1\rangle}}) - \mathbb{E}\left(X(0)X(h)^{\langle p-1\rangle}\right)\right) = 0 \quad a.s.$$

for each $\gamma \in (p/\beta, 1)$, where $\mathbf{m} \to \infty$ is to be understood componentwise, see Móricz et al. (2008, Corollary 2). Therefore, the estimators (15) and (16) are strongly consistent with rates of convergence $O(1/|\mathbf{m}|^{1-\gamma})$ and $O(1/\operatorname{card}(D_h(\mathbf{m}))^{1-\gamma})$. We have almost surely

$$\hat{\kappa}(h) - \kappa(h) = \frac{\left(\widehat{\mathbb{E}|X(0)|^p}\right)^{\alpha/p-1} \mathbb{E}\,(\widehat{X(0)X(h)^{\langle p-1\rangle}})}{c_\alpha(p)^\alpha} - \kappa(h) = O\left(\frac{1}{|\mathbf{m}|^{1-\gamma}}\right), \quad \mathbf{m} \to \infty.$$

Notice that we can obtain a rate of convergence of $O(1/|\mathbf{m}|^{1-\gamma})$ for any $\gamma \in (1/\alpha, 1)$ by choosing p and β appropriately. The sample size is given by $l = |\mathbf{m}|$. It is straightforward to construct a sequence $\{\varepsilon_l\}_{l\in\mathbb{N}}$ which satisfies the conditions of Theorem 1, for example $\varepsilon_l = 1/(l^{(1-\gamma)/2})$.

(b) The estimator $\hat{f}_{(1,r+1)}$ based on (10) is obtained by finding the root of the function

$$g(x) = \hat{f}_{(1,q)} x^{\langle \alpha-1\rangle} + \hat{f}_{(1,q)}^{\langle \alpha-1\rangle} x - \hat{\kappa}(h_{(1,r+1)}) + \hat{a}(r).$$

Therefore, it is essential to have a reliable estimate $\hat{f}_{(1,q)}$ if the theoretical value $f_{(1,q)}$ is equal to zero. To account for this and stabilize the estimation procedure, we multiply the raw estimators in (6)–(8) with the corresponding indicator functions.

(c) In (6)–(8), one has to decide whether to take the positive or negative sign in the definition of $\hat{f}_{(i,j)}$. Assume that $\hat{f}^+_{(1,1)}, \ldots, \hat{f}^+_{(n,n)}$ and $\hat{f}^-_{(1,1)}, \ldots, \hat{f}^-_{(n,n)}$ are the estimates based on the uniform choice of the positive and negative sign, respectively. Then (10), (11) and (12) imply $\hat{f}^+_{(i,j)} = -\hat{f}^-_{(i,j)}$ for $i, j = 1, \ldots, m$. Let f^+ and f^- denote the corresponding kernel functions and X^+ and X^- the corresponding random fields. If X is symmetric, then the finite-dimensional distributions of X^- are equal to the ones of X^+ due to Samorodnitsky and Taqqu (1994, formula (3.2.4)), i. e. it does not matter whether to take the positive or negative value in (6)–(8). If X is non-symmetric, then $\beta_{X^+(t)} = -\beta_{X^-(t)}$, where $\beta_{X^\pm(t)}$ denotes the skewness parameter of $X^\pm(t)$, $t \in \mathbb{R}^d$, cf. Samorodnitsky and Taqqu (1994, Proposition 3.2.2). Also, the random field (1) is stationary since it is a moving average with Lebesgue control measure. Thus, one can base the choice of a positive or negative sign in (6)–(8) on an estimate of the skewness of the marginal distributions as follows.

Assume that the skewness intensity of the stable random measure in (1) is constant and equal to β. Then

$$\beta_{X(t)} = \beta \frac{\int_{\mathbb{R}^d} f(x)^{\langle\alpha\rangle} dx}{\int_{\mathbb{R}^d} |f(x)|^\alpha dx},$$

see Samorodnitsky and Taqqu (1994, Proposition 3.4.1). Plugging the estimates $\hat{\beta}_{X(t)}$ and $\hat{f}^{\pm}$ of $\beta_{X(t)}$ and f in the last equation, we obtain for $\hat{f}^{+}$ and $\hat{f}^{-}$ a unique $\hat{\beta}^{+}$ and $\hat{\beta}^{-}$ (with $\hat{\beta}^{-} = -\hat{\beta}^{+}$) such that the marginal distributions of X have the appropriate skewness. For modeling purposes, both pairs $(\hat{f}^{+}, \hat{\beta}^{+})$ and $(\hat{f}^{-}, \hat{\beta}^{-})$ are reasonable choices because they yield the same finite-dimensional distributions of the estimated stable random field, cf. the following Lemma 1. A method to estimate the skewness parameter of an α-stable distribution is presented in McCulloch (1986).

Lemma 1. *Let $1 < \alpha < 2$ and $X^{\pm} = (\int_{\mathbb{R}^d} f_1^{\pm}(x)M^{\pm}(dx), \ldots, \int_{\mathbb{R}^d} f_n^{\pm}(x)M^{\pm}(dx))^{\mathsf{T}}$, $d, n \in \mathbb{N}$, be two stable random vectors, where $f^{-}(x) = -f^{+}(x)$ for all $x \in \mathbb{R}^d$ and $M^{\pm}$ are stable random measures with Lebesgue control measure and constant skewness intensities $\beta^{\pm}$ with $\beta^{-} = -\beta^{+}$. Then $X^{+} \stackrel{d}{=} X^{-}$.*

Proof. The characteristic function of $X^{\pm}$ is given by

$$\varphi(z) = \exp\left\{-\int_{\mathbb{S}^{d-1}} |\langle z, s\rangle|^{\alpha} \left(1 - i\operatorname{sign}(\langle z, s\rangle) \tan\frac{\pi\alpha}{2}\right)\Gamma^{\pm}(ds)\right\}, \quad z \in \mathbb{R}^d,$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and $\Gamma^{\pm}$ are finite measures on $\mathbb{S}^{d-1}$, see Samorodnitsky and Taqqu (1994, Theorem 2.3.1). For given α, the measures $\Gamma^{\pm}$ completely determine the distribution of $X^{\pm}$. By Samorodnitsky and Taqqu (1994, formula (3.2.4)), we have for each Borel set A of $\mathbb{S}^{d-1}$

$$\Gamma^{\pm}(A) = \frac{1+\beta^{\pm}}{2} m(g_{\pm}^{-1}(A)) + \frac{1-\beta^{\pm}}{2} m(g_{\pm}^{-1}(-A)),$$

where

$$\begin{aligned}
m(dx) &= \left(\sum_{i=1}^{n} f_i^{\pm}(x)^2\right)^{\alpha/2} dx, \quad E = \left\{x \in \mathbb{R}^d : \sum_{i=1}^{n} f_i^{\pm}(x)^2 > 0\right\}, \\
g_i^{\pm}(x) &= \frac{f_i^{\pm}(x)}{(\sum_{i=1}^{n} f_i^{\pm}(x)^2)^{1/2}}, \quad i = 1, \ldots, n,\ x \in E, \quad g_{\pm}^{-1}(A) = \left\{x \in E : (g_1^{\pm}(x), \ldots, g_n^{\pm}(x))^{\mathsf{T}} \in A\right\}.
\end{aligned}$$

Since $f^{-}(x) = -f^{+}(x)$ for all $x \in \mathbb{R}^d$, we have $g_{+}^{-1}(A) = g_{-}^{-1}(-A)$ which implies together with $\beta^{-} = -\beta^{+}$

$$\begin{aligned}
\Gamma^{+}(A) &= \frac{1+\beta^{+}}{2} \int_{g_{+}^{-1}(A)} \left(\sum_{i=1}^{n} f_i^{+}(x)^2\right)^{\alpha/2} dx + \frac{1-\beta^{+}}{2} \int_{g_{+}^{-1}(-A)} \left(\sum_{i=1}^{n} f_i^{+}(x)^2\right)^{\alpha/2} dx \\
&= \frac{1-\beta^{-}}{2} \int_{g_{-}^{-1}(-A)} \left(\sum_{i=1}^{n} f_i^{-}(x)^2\right)^{\alpha/2} dx + \frac{1+\beta^{-}}{2} \int_{g_{-}^{-1}(A)} \left(\sum_{i=1}^{n} f_i^{-}(x)^2\right)^{\alpha/2} dx = \Gamma^{-}(A).
\end{aligned}$$

□

3. Practical issues and open problems

Let $d = 2$ and $\Delta_{\mathbf{i}}$, $\mathbf{1} \le \mathbf{i} \le \mathbf{n}$, be squares with side length 1. We choose $n = 5$ and simulate a symmetric 1.8-stable random field with the kernel f given in the left plot of Figure 5 on the grid $[1, 500]^2 \cap \mathbb{N}^2$. We advice that in practical applications, the side length of the observation window is at least two times the diameter of the support of f in order to reduce boundary effects with the estimation of the covariation function. We use Remark 2 and $p = 1.4$ to estimate the covariation function from the realization of the stable random field.

The choice of ε_l mainly determines the quality of the kernel function estimation. By Theorem 1, we only know how the sequence $\{\varepsilon_l\}_{l\in\mathbb{N}}$ has to behave as $l \to \infty$. However, for some fixed $l \in \mathbb{N}$, it is not clear which value we should take. As one can see in Figure 5, we completely fail to retrieve the kernel if ε_l is too small ($\varepsilon_l = 0.01$). The estimate also differs evidently from the theoretical kernel if ε_l is too large ($\varepsilon_l = 1.3$). An appropriate choice for ε_l is 1.086 for which the kernel is estimated very well.

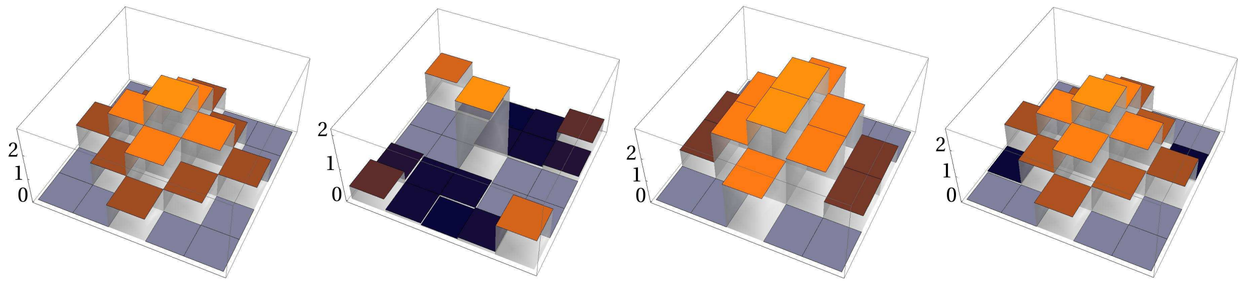

Figure 5: The theoretical kernel f (first plot) and the estimated kernel with $\varepsilon_l = 0.01$ (second plot), $\varepsilon_l = 1.3$ (third plot) and $\varepsilon_l = 1.086$ (forth plot).

In applications, one possibility to handle this problem is to look at the estimated covariation function and set a plausible value by hand. Another possibility is to use different choices of ε_l and take the resulting kernel which looks most reasonable.

We now present an approach to obtain a reasonable choice of ε_l. The intuition of the method is as follows. It holds $\kappa(0) = \int_{\mathbb{R}^d} |f(x)|^\alpha dx$. We transform the subgraph of the function $|f|^\alpha$ to a cuboid $C(f)$ which has the same volume $\nu_d(C(f)) = \int_{\mathbb{R}^d} |f(x)|^\alpha dx$ as the volume below the graph of $|f|^\alpha$ and the same support $K = \cup_{\mathbf{1 \le i \le n}} \Delta_{\mathbf{i}}$, see Figure 6. The height δ of the cuboid is equal to the mean height of $|f|^\alpha$. Let $K = \operatorname{supp}(f)$ be the support of f. Then we have

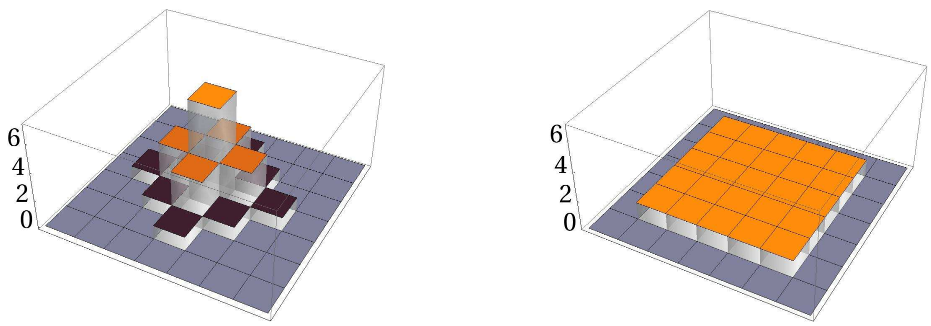

Figure 6: The function $|f|^\alpha$ (left plot) and the corresponding cuboid $C(f)$ (right plot).

$\kappa(0) = \delta \nu_d(K)$ and therefore $\delta = \kappa(0)/\nu_d(K)$. The last formula yields the practical choice of the order of magnitude of ε_l, namely

$$\varepsilon_l \approx \frac{\hat{\kappa}(0)}{\nu_d(K)}.$$

In the numerical example, we have $\nu_2(K) = 5 \cdot 5 = 25$ and $\hat{\kappa}(0) = 27.152$ which implies $\varepsilon_l = 1.086$. In Figure 5, we see that this value for ε_l retrieves the actual kernel very well.

We now address some open problems. If X is a symmetric stable random field, the covariation function can be estimated via (14). For non-symmetric stable random fields, the use of (13) to estimate the covariation function is not advisable because the quantity $(X, Y)_\alpha$ involves the spectral measure of a stable random vector. In this case, it is more straightforward to consider directly the formula $[X, Y]_\alpha = \int_{\mathbb{S}^1} s_1 s_2^{\langle \alpha - 1 \rangle} \Gamma(ds)$ and estimate it via an estimation of the spectral measure. Estimation methods for the spectral measure can be found in Nolan et al. (2001); Pivato and Seco (2003). However, this approach is not so efficient and better methods to estimate the covariation function may be developed.

In Remark 2(c), we addressed the problem of choosing the correct sign in the definition of $\hat{f}_{(i,j)}$ and presented a solution when the skewness intensity of the stable random measure is constant. For stable random measures with non-constant skewness intensity, it is not clear yet how to fix the sign.

Finally, the choice of the cut-off parameter ε_l might be based on a more systematic approach by considering techniques which are applied in kernel density estimation to obtain the optimal band width parameter.

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