

# Set Reconstruction by Voronoi cells

Matthias Reitzner\*, Evgeny Spodarev†, Dmitry Zaporozhets‡

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## Abstract

For a Borel set  $A$  and a homogeneous Poisson point process  $\eta$  in  $\mathbb{R}^d$  of intensity  $\lambda > 0$ , define the Poisson–Voronoi approximation  $A_\eta$  of  $A$  as a union of all Voronoi cells with nuclei from  $\eta$  lying in  $A$ . If  $A$  has a finite volume and perimeter we find an exact asymptotic of  $\mathbb{E} \text{Vol}(A \Delta A_\eta)$  as  $\lambda \rightarrow \infty$  where  $\text{Vol}$  is the Lebesgue measure. Estimates for all moments of  $\text{Vol}(A_\eta)$  and  $\text{Vol}(A \Delta A_\eta)$  together with their asymptotics for large  $\lambda$  are obtained as well.

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## 1 Introduction

Let  $A$  be a Borel set in  $\mathbb{R}^d$  and  $\eta$  be a Poisson point process in  $\mathbb{R}^d$ . Assume that we observe  $\eta$  and the only information about  $A$  at our disposal is which points of  $\eta$  lie in  $A$ , i.e., we have the partition of the process  $\eta$  into  $\eta \cap A$  and  $\eta \setminus A$ . We try to reconstruct the set  $A$  just by the information contained in these two point sets. For that we approximate  $A$  by the set  $A_\eta$  of all points in  $\mathbb{R}^d$  which are closer to  $\eta \cap A$  than to  $\eta \setminus A$ .

More formally, let  $\eta$  be a homogeneous Poisson point process of intensity  $\lambda > 0$ , and denote by  $v_\eta(x) = \{z \in \mathbb{R}^d : \|z - x\| \leq \|z - y\| \text{ for all } y \in \eta\}$  the Voronoi cell generated

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\*Postal address: Institut für Mathematik, Universität Osnabrück, 49069 Osnabrück, Germany. Email: matthias.reitzner@uni-osnabrueck.de

†Partially supported by RFBR-DFG (09-0191331) and DFG (436 RUS 113/962/0-1 R) grants. Postal address: Institut für Stochastik, Universität Ulm, 89069 Ulm, Germany. Email: evgeny.spodarev@uni-ulm.de

‡Partially supported by RFBR (10-01-00242), NSh-4472.2010.1, RFBR-DFG (09-0191331) and DFG (436 RUS 113/962/0-1 R) grants. Postal address: St.Petersburg Department of V.A.Steklov Mathematical Institute, Fontanka 27, 191023 St.Petersburg, Russia. Email: zap1979@gmail.com

by  $\eta$  with nucleus  $x \in \eta$ . Then the set  $A_\eta$  is just the union of the Poisson–Voronoi cells with nuclei lying in  $A$ , i.e.,

$$A_\eta = \bigcup_{x \in \eta \cap A} v_\eta(x).$$

We call this set the *Poisson–Voronoi approximation* of the set  $A$ . It was first introduced by Khmaladze and Toronjadze in [8]. They proposed  $A_\eta$  to be an estimator for  $A$  when  $\lambda$  is large (potential applications are listed in [7, Section 1]). They conjectured that for arbitrary bounded Borel set  $A \subset \mathbb{R}^d$ ,  $d \geq 1$ , it holds

$$\begin{aligned} \text{Vol}(A_\eta) &\rightarrow \text{Vol}(A), & \lambda &\rightarrow \infty, \\ \text{Vol}(A \Delta A_\eta) &\rightarrow 0, & \lambda &\rightarrow \infty, \end{aligned} \tag{1}$$

almost surely, where  $\text{Vol}(\cdot)$  stands for the Lebesgue measure (volume) and  $\Delta$  is the operation of the symmetric difference of sets. This conjecture was proved in [8] for  $d = 1$ . The case of general  $d$  was treated by Einmahl and E. V. Khmaladze in [4] with some technical assumption on the boundary of  $A$ , and then generalized by Penrose in [11] to an arbitrary bounded Borel set  $A$ .

It can be easily shown (see Section 3 for details) that for any Borel set  $A$  it holds

$$\mathbb{E} \text{Vol}(A_\eta) = \text{Vol}(A).$$

Thus  $\text{Vol}(A_\eta)$  is an unbiased estimator for the volume of  $A$ . In this paper we also consider the  $n$ -th moment of  $\text{Vol}(A_\eta)$  and approximate it by the  $n$ -th degree of the volume of the original set  $\text{Vol}^n(A)$  asymptotically as  $\lambda \rightarrow \infty$  (Theorem 2.2). For the case when  $n = 2$  and  $A$  is a convex compact, similar estimates were obtained in [7].

It might be suggested from (1) that

$$\mathbb{E} \text{Vol}(A \Delta A_\eta) \rightarrow 0, \quad \lambda \rightarrow \infty, \tag{2}$$

although it is not a direct corollary. The more interesting problem is to find an exact asymptotic of  $\mathbb{E} \text{Vol}(A \Delta A_\eta)$ . Initially it was considered by Heveling and Reitzner in [7]. They proved that for any compact convex set  $A$  with surface area  $S(A)$  it holds

$$\mathbb{E} \text{Vol}(A \Delta A_\eta) = c_d \cdot S(A) \cdot \lambda^{-1/d} (1 + O(\lambda^{-1/d})), \quad \lambda \rightarrow \infty,$$

where the constant  $c_d$  independent of  $\lambda$  and  $A$  was calculated by them in an explicit form (see Section 2 for details). Here we obtain a similar asymptotic formula (Theorem 2.1) for a much wider class of sets. Namely, we consider Borel sets with finite volume  $\text{Vol}(A)$  and perimeter  $\text{Per}(A)$  (see Section 3 for the precise definition). Our methods are completely different from those of Heveling and Reitzner. The key observations are the connection between the Poisson–Voronoi approximation and the covariogram of  $A$ , and the connection

between the covariogram and the perimeter of a set recently established by Galerne [5]. As a by-product of our calculations, we prove that (2) holds for any Borel set  $A$  with finite volume (Corollary 4.1).

We also consider higher moments of  $\text{Vol}(A\Delta A_\eta)$ . For arbitrary Borel set  $A$  we approximate  $\mathbb{E} \text{Vol}^n(A\Delta A_\eta)$  by the  $n$ -th degree of  $\mathbb{E} \text{Vol}(A\Delta A_\eta)$  asymptotically as  $\lambda \rightarrow \infty$  (Theorem 2.3). Thus, assuming that  $\text{Vol}(A), \text{Per}(A) < \infty$  and using the asymptotic for  $\mathbb{E} \text{Vol}(A\Delta A_\eta)$  from Theorem 2.1, we obtain the asymptotic for  $\mathbb{E} \text{Vol}^n(A\Delta A_\eta)$  (Corollary 2.1).

The paper is organized as follows. Our main results are stated in the next section. In Section 3, we introduce the necessary background and notation, in particular the perimeter and the covariogram of a set  $A$ . Proofs are given in Section 4.

## 2 Main results

Our first result yields the asymptotic of the average volume of  $A\Delta A_\eta$  with increasing intensity  $\lambda$ . To formulate it, we need to define a notion of perimeter of a Borel set. The definition is somewhat technical, so we postpone it till Section 3. If  $A$  is a compact set with Lipschitz boundary (e.g. a convex body), then  $\text{Per}(A)$  equals the  $(d-1)$ -dimensional Hausdorff measure  $\mathcal{H}_{d-1}(\partial A)$  of the boundary  $\partial A$  of  $A$ . In general case it holds  $\text{Per}(A) \leq \mathcal{H}_{d-1}(\partial A)$  (see, e.g. [1, Proposition 3.62]). Therefore,  $\text{Per}(A)$  could be replaced by  $\mathcal{H}_{d-1}(\partial A)$  in the assumptions of the theorem.

**Theorem 2.1.** *If  $A \subset \mathbb{R}^d$  is a Borel set with  $\text{Vol}(A) < \infty$  and  $\text{Per}(A) < \infty$ , then*

$$\mathbb{E} \text{Vol}(A\Delta A_\eta) = c_d \cdot \text{Per}(A) \cdot \lambda^{-1/d} (1 + o(1)), \quad \lambda \rightarrow \infty, \quad (3)$$

where  $c_d = 2d^{-2} \Gamma(1/d) \kappa_{d-1} \kappa_d^{-1-1/d}$  and  $\kappa_n$  is the volume of the unit  $n$ -dimensional ball.

The probabilistic intuition behind this asymptotic is the following. The set difference  $A\Delta A_\eta$  behaves asymptotically as a very small tube neighbourhood of the boundary  $\partial A$  formed out of the Poisson–Voronoi cells with nuclei lying almost on  $\partial A$ . Since the volume of a typical Poisson–Voronoi cell is  $\lambda^{-1}$ , its diameter has the order  $\lambda^{-1/d}$ , and so the volume of this tube neighborhood has the order  $\text{Per}(A) \lambda^{-1/d}$ .

In the following, saying that some inequality holds asymptotically as  $\lambda \rightarrow \infty$ , we mean that it holds for sufficiently large  $\lambda \geq \lambda_0$ . The choice of  $\lambda_0$  might depend on  $A$ . Thus, all estimates are not uniform with respect to  $A$  (including those of Theorem 2.1).

**Theorem 2.2.** *If  $A \subset \mathbb{R}^d$  is a Borel set with  $\text{Vol}(A) < \infty$ , then*

$$\left| \mathbb{E} \text{Vol}^n(A_\eta) - \text{Vol}^n(A) \right| \leq C_{n,d} \cdot \text{Vol}^{n-1}(A) \cdot \lambda^{-1}, \quad \lambda \rightarrow \infty,$$

where  $C_{n,d}$  is some constant independent of  $\lambda$  and  $A$ .

**Remark 2.1.** *In fact, we show that the following non-asymptotic inequality holds: for any  $\lambda > 0$*

$$\left| \mathbb{E} \text{Vol}^n(A_\eta) - \text{Vol}^n(A) \right| \leq C_{n,d} \cdot \sum_{k=1}^{n-1} \text{Vol}^{n-k}(A) \cdot \lambda^{-k}.$$

**Theorem 2.3.** *If  $A \subset \mathbb{R}^d$  is a Borel set with  $\text{Vol}(A) < \infty$  and  $\text{Per}(A) < \infty$ , then*

$$\left| \mathbb{E} \text{Vol}^n(A\Delta A_\eta) - (\mathbb{E} \text{Vol}(A\Delta A_\eta))^n \right| \leq C'_{n,d} \cdot \text{Per}(A)^{n-1} \cdot \lambda^{-1-(n-1)/d}, \quad \lambda \rightarrow \infty,$$

where  $C'_{n,d}$  is some constant independent of  $\lambda$  and  $A$ .

**Remark 2.2.** *We conjecture that the following limit theorems can be proven by the method of moments (see e.g. [3, Theorems 30.1, 30.2]):*

$$\lambda^{1/2(1+1/d)} (\text{Vol}(A_\eta) - \text{Vol}(A)) \rightarrow N(0, \sigma_1 \text{Per}(A)), \quad (4)$$

$$\lambda^{1/2(1+1/d)} \left( \text{Vol}(A\Delta A_\eta) - c_d \text{Per}(A) \lambda^{-1/d} \right) \rightarrow N(0, \sigma_2 \text{Per}(A))$$

in distribution as  $\lambda \rightarrow \infty$ ,  $\sigma_1, \sigma_2 > 0$ .

Recently (4) was proved by Schulte [12] for *convex* sets  $A$  using a central limit theorem for Wiener-Itó chaos expansions. In his Remark 4 he points out that the result can be extended to all sets where the volume of a small tube neighbourhood  $B(\partial A)$  of  $\partial A$  can be bounded in a nice way. Yet the general conjecture seems to be open.

**Corollary 2.1.** *If  $A \subset \mathbb{R}^d$  is a Borel set with  $\text{Vol}(A) < \infty$  and  $\text{Per}(A) < \infty$ , then*

$$\mathbb{E} \text{Vol}^n(A\Delta A_\eta) = (\mathbb{E} \text{Vol}(A\Delta A_\eta))^n (1 + O(\lambda^{-1+1/d})), \quad \lambda \rightarrow \infty,$$

and for  $d \geq 2$

$$\mathbb{E} \text{Vol}^n(A\Delta A_\eta) = (c_d \text{Per}(A))^n \lambda^{-n/d} (1 + o(1)), \quad \lambda \rightarrow \infty.$$

The asymptotic order of the variance of  $A_\eta$  and  $A\Delta A_\eta$  as  $\lambda \rightarrow \infty$  was first studied in [7] for convex sets  $A$ . We extend that results to arbitrary Borel sets.

**Corollary 2.2.** *If  $A \subset \mathbb{R}^d$  is a Borel set with  $\text{Vol}(A) < \infty$  and  $\text{Per}(A) < \infty$ , then*

$$\text{Var} \text{Vol}(A_\eta) \leq C_d \cdot \text{Per}(A) \cdot \lambda^{-1-1/d}, \quad \lambda \rightarrow \infty,$$

and

$$\text{Var} \text{Vol}(A\Delta A_\eta) \leq C_d \cdot \text{Per}(A) \cdot \lambda^{-1-1/d}, \quad \lambda \rightarrow \infty,$$

where  $C_d$  is some constant independent of  $\lambda$  and  $A$ .

The second inequality follows immediately from Theorem 2.3. The first inequality will be proved in Section 4.2.

The probabilistic heuristic explaining the asymptotic behavior of the variances is the following. Since  $A\Delta A_\eta$  is asymptotically a very small tube neighbourhood  $B(\partial A)$  of  $\partial A$  consisting of parts  $\tilde{v}_\eta(x)$  of almost independent Poisson–Voronoi cells  $v_\eta(x)$  with nuclei  $x \in B(\partial A)$  we may use the formula for the variance of the compound Poisson distribution:

$$\text{Var Vol}(A\Delta A_\eta) = \text{Var} \left( \sum_{x \in \eta \cap B(\partial A)} \text{Vol}(\tilde{v}_\eta(x)) \right) \approx \text{Var} \left( \sum_{i=1}^N Y_i \right)$$

where random variables  $Y_i \stackrel{d}{=} \text{Vol}(\tilde{v}_\eta(x))$  are *i.i.d.* and

$$N \stackrel{d}{=} \text{card}(\eta \cap B(\partial A)) \sim \text{Pois}(\lambda \text{Vol}(B(\partial A)))$$

is independent of  $Y_i$ . Here  $\stackrel{d}{=}$  means the equality in distribution and  $\text{card}(B)$  is the cardinality of a set  $B$ . Then

$$\begin{aligned} \text{Var} \left( \sum_{i=1}^N Y_i \right) &= \mathbb{E} N \text{Var} Y_1 + \text{Var} N (\mathbb{E} Y_1)^2 = \lambda \text{Vol}(B(\partial A)) \mathbb{E} Y_1^2 \\ &\leq \lambda \text{Vol}(B(\partial A)) (\mathbb{E} \text{Vol}(v_\eta(x)))^2 = O \left( \lambda \text{Per}(A) \lambda^{-1/d} \lambda^{-2} \right) = \text{Per}(A) O \left( \lambda^{-1-1/d} \right) \end{aligned}$$

since  $\tilde{v}_\eta(x) \subset v_\eta(x)$  for any  $x$ , the second moment of the volume of a typical Poisson–Voronoi cell is of order  $\lambda^{-2}$  and the volume of  $B(\partial A)$  is of order  $\text{Per}(A) \lambda^{-1/d}$ .

The results of Corollary 2.2 can also be obtained by using the Poincaré inequality which gives an upper bound on the variance of a functional of a Poisson point process. Let  $\mathcal{N}$  be the set of all locally finite configurations on  $\mathbb{R}^d$ . Consider a nonnegative measurable function  $F : \mathcal{N} \rightarrow \mathbb{R}$ . If  $\mathbb{E} F^2(\eta) < \infty$ , then

$$\text{Var} F(\eta) \leq \lambda \mathbb{E} \int_{\mathbb{R}^d} (F(\eta \cup \{y\}) - F(\eta))^2 dy, \quad (5)$$

where we added a point  $y$  to the Poisson point process  $\eta$ . Putting  $F(\eta) = \text{Vol}(A_\eta)$  in (5), we get

$$\text{Var Vol}(A_\eta) \leq \lambda \int_{\mathbb{R}^d} \mathbb{E} (\text{Vol}(A_{\eta \cup \{y\}}) - \text{Vol}(A_\eta))^2 dy,$$

where the right-hand side can be estimated from above to get the upper bound in Corollary 2.2. The reasoning for the symmetric difference  $A\Delta A_\eta$  is similar.

In full generality, inequality (5) was proved by Wu [15]. As was shown by Last and Penrose [9, Theorem 1.2], it is a consequence of an even more general inequality following from the Fock space representation of Poisson point processes.

### 3 Preliminaries

For basic facts from integral geometry, stochastic geometry and Voronoi tessellations which are not explained in the following, we refer the reader to [13], [14], and [10].

Define the perimeter of a Borel set  $A$  as

$$\text{Per}(A) = \sup \left\{ \int_A \text{div } \varphi(x) dx : \varphi \in \mathcal{C}_c^1(\mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\},$$

cf. [1], where

$$\text{div } \varphi(x) = \sum_{i=1}^d \frac{\partial \varphi_i}{\partial x_i} \quad \text{and} \quad \|\varphi\|_\infty = \max_{i=1, \dots, d} \sup_{x \in \mathbb{R}^d} |\varphi_i(x)|$$

for  $\varphi = (\varphi_1, \dots, \varphi_d)$ . The class  $\mathcal{C}_c^1(\mathbb{R}^d)$  consists of all continuously differentiable vector-valued functions from  $\mathbb{R}^d$  to  $\mathbb{R}^d$  with compact support.

Let  $A$  be a Borel set with finite volume. Then

$$g_A(x) = \text{Vol}((A+x) \cap A), \quad x \in \mathbb{R}^d,$$

is a covariogram of  $A$ . For the history on the covariogram problem see the references in [5] and also the recent breakthrough by Averkov and Bianchi [2].

In the proof of Theorem 2.1 we use the result obtained by Galerne in [5, Theorem 14]. The following assertions are equivalent:

- (a)  $\text{Per}(A) < \infty$ ;
- (b) there exists a finite limit

$$\lim_{r \rightarrow +0} \frac{g_A(ru) - g_A(0)}{r} = \frac{\partial g_A}{\partial u}(0) \tag{6}$$

for all  $u \in \mathbb{S}^{d-1}$ ;

- (c)  $g_A$  is Lipschitz.

In addition, the Lipschitz constant of  $g_A$  satisfies

$$\text{Lip}(g_A) \leq \frac{1}{2} \text{Per}(A) \tag{7}$$

and it holds

$$\int_{\mathbb{S}^{d-1}} \frac{\partial g_A}{\partial u}(0) \mathcal{H}_{d-1}(du) = -\kappa_{d-1} \text{Per}(A). \tag{8}$$

Another tool we need is the refined Campbell–Mecke formula for stationary point processes (cf. e.g. [14]). Using Slivnyak’s theorem, we give its particular case for the Poisson point process.

As above, let  $\eta$  be a homogeneous Poisson point process of intensity  $\lambda > 0$ , and  $\mathcal{N}$  be the set of all locally finite point configurations on  $\mathbb{R}^d$ . Consider a nonnegative measurable function  $f : \mathcal{N} \times (\mathbb{R}^d)^m \rightarrow \mathbb{R}$ . Then

$$\mathbb{E} \sum_{(y_1, \dots, y_m) \in \eta_{\neq}^m} F(\eta, y_1, \dots, y_m) = \lambda^m \int_{(\mathbb{R}^d)^m} \mathbb{E} F(\eta \cup \bar{y}_m, y_1, \dots, y_m) dy_1 \dots dy_m, \quad (9)$$

where  $\eta_{\neq}^m$  denotes the set of all  $m$ -tuples of pair-wise distinct points from  $\eta$ , and  $\eta \cup \bar{y}_m$  is the process  $\eta$  with added point set  $\bar{y}_m = \{y_1, \dots, y_m\}$ .

As a simple corollary we get two identities which are crucial for us in the sequel.

**Proposition 3.1.** *If  $A \subset \mathbb{R}^d$  is a Borel set with  $\text{Vol}(A) < \infty$ , then*

$$\mathbb{E} \text{Vol}(A_\eta) = \lambda \int_{\mathbb{R}^d} \int_A e^{-\lambda \kappa_d \|y-x\|^d} dy dx = \text{Vol}(A), \quad (10)$$

$$\mathbb{E} \text{Vol}(A \Delta A_\eta) = 2\lambda \int_{\mathbb{R}^d \setminus A} \int_A e^{-\lambda \kappa_d \|y-x\|^d} dy dx. \quad (11)$$

*Proof.* By Fubini’s theorem and the Slivnyak-Mecke formula (9), we have

$$\begin{aligned} \mathbb{E} \text{Vol}(A_\eta) &= \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}(x \in A_\eta) dx = \int_{\mathbb{R}^d} \mathbb{E} \sum_{y \in \eta \cap A} \mathbf{1}(x \in v_\eta(y)) dx \\ &= \lambda \int_{\mathbb{R}^d} \int_A \mathbb{P}(x \in v_{\eta \cup \{y\}}(y)) dy dx = \lambda \int_{\mathbb{R}^d} \int_A e^{-\lambda \kappa_d \|x-y\|^d} dy dx. \end{aligned}$$

Similarly, we obtain

$$\mathbb{E} \text{Vol}(A \setminus A_\eta) = \lambda \int_A \int_{\mathbb{R}^d \setminus A} e^{-\lambda \kappa_d \|x-y\|^d} dy dx$$

and

$$\mathbb{E} \text{Vol}(A_\eta \setminus A) = \lambda \int_{\mathbb{R}^d \setminus A} \int_A e^{-\lambda \kappa_d \|x-y\|^d} dy dx.$$

By definition  $\text{Vol}(A\Delta A_\eta) = \text{Vol}(A \setminus A_\eta) + \text{Vol}(A_\eta \setminus A)$  which completes the proof of (11). To prove the second part of (10), one has to apply Fubini's theorem and then use the formula

$$\int_{\mathbb{R}^d} e^{-c\|x-y\|^d} dx = \frac{\kappa_d}{c}, \quad c > 0, \quad (12)$$

which could be easily proved by introducing spherical coordinates.  $\square$

Notice that we have also proved that

$$\mathbb{E} \text{Vol}(A \setminus A_\eta) = \mathbb{E} \text{Vol}(A_\eta \setminus A).$$

However,  $\text{Vol}(A \setminus A_\eta)$  and  $\text{Vol}(A_\eta \setminus A)$  are not equidistributed since the first random variable is bounded, and the second is not. As a direct colollary of the identity (10) we get

$$\text{Var} \text{Vol}(A_\eta) = \mathbb{E} (\text{Vol}(A \setminus A_\eta) - \text{Vol}(A_\eta \setminus A))^2, \quad (13)$$

which we shall use in the following.

## 4 Proofs

### 4.1 Asymptotics of the mean volume of the symmetric difference

In this section we give the proof of Theorem 2.1. The key step to prove it is the following relation between the Poisson–Voronoi approximation and the covariogram of a set  $A$ .

**Lemma 4.1.** *Let  $g_A(x)$  be the covariogram of a Borel set  $A$  with  $\text{Vol}(A) < \infty$ . Then*

$$\mathbb{E} \text{Vol}(A\Delta A_\eta) = -2 \int_0^\infty r^{d-1} e^{-\kappa_d r^d} \tilde{g}_A(\lambda^{-1/d} r) dr, \quad (14)$$

where

$$\tilde{g}_A(r) = \int_{\mathbb{S}^{d-1}} (g_A(ru) - g_A(0)) \mathcal{H}_{d-1}(du). \quad (15)$$

*Proof.* Replacing  $y$  in (11) by  $x - \lambda^{-1/d} z$  we get

$$\begin{aligned} \mathbb{E} \text{Vol}(A\Delta A_\eta) &= 2\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\lambda \kappa_d \|y-x\|^d} \mathbf{1}\{y \in A, x \in A^c\} dy dx \\ &= 2 \int_{\mathbb{R}^d} e^{-\kappa_d \|z\|^d} \int_{\mathbb{R}^d} \mathbf{1}\{x \in (A + \lambda^{-1/d} z) \cap A^c\} dz dx \\ &= 2 \int_{\mathbb{R}^d} e^{-\kappa_d \|z\|^d} \text{Vol}((A + \lambda^{-1/d} z) \cap A^c) dz. \end{aligned}$$

By the definition of the covariogram  $\text{Vol}((A + \lambda^{-1/d}z) \cap A^c) = g_A(0) - g_A(\lambda^{-1/d}z)$ . We introduce spherical coordinates  $z = ru$ , where  $r \in \mathbb{R}^+$  and  $u \in \mathbb{S}^{d-1}$ . This yields

$$\mathbb{E}\text{Vol}(A\Delta A_\eta) = -2 \int_0^\infty r^{d-1} e^{-\kappa_d r^d} \left[ \int_{\mathbf{S}^{d-1}} \left( g_A(\lambda^{-1/d}ru) - g_A(0) \right) \mathcal{H}_{d-1}(du) \right] dr.$$

□

**Corollary 4.1.** *For any measurable  $A$  with  $\text{Vol}(A) < \infty$  it holds*

$$\mathbb{E}\text{Vol}(A\Delta A_\eta) \rightarrow 0, \quad \lambda \rightarrow \infty.$$

*Proof.* It immediately follows from (14) and the continuity of the set covariogram. □

*Proof of Theorem 2.1.* Using Lemma 4.1 and substituting  $t$  for  $\kappa_d r^d$  we obtain

$$\mathbb{E}\text{Vol}(A\Delta A_\eta) = -\frac{2}{d\kappa_d} \int_0^\infty e^{-t} \tilde{g}_A \left( (\lambda\kappa_d)^{-1/d} t^{1/d} \right) dt.$$

It follows from (7) and the definition of  $\tilde{g}_A$  that

$$|\tilde{g}_A(r)| \leq \frac{1}{2} \mathcal{H}_{d-1}(\mathbf{S}^{d-1}) \text{Per}(A)r. \quad (16)$$

Therefore, Lebesgue's Dominated Convergence Theorem and equations (6), (8) yield

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \mathbb{E}\text{Vol}(A\Delta A_\eta) \lambda^{1/d} &= -\frac{2}{d} \kappa_d^{-1-1/d} \lim_{\lambda \rightarrow \infty} \int_0^\infty e^{-t} t^{1/d} \frac{\tilde{g}_A((\lambda\kappa_d)^{-1/d} t^{1/d})}{(\lambda\kappa_d)^{-1/d} t^{1/d}} dt \\ &= -\frac{2}{d} \kappa_d^{-1-1/d} \int_0^\infty e^{-t} t^{1/d} dt \int_{\mathbf{S}^{d-1}} \frac{\partial g_A}{\partial u}(0) \mathcal{H}_{d-1}(du) \\ &= \frac{2}{d} \kappa_{d-1} \kappa_d^{-1-1/d} \text{Per}(A) \int_0^\infty e^{-t} t^{1/d} dt = \frac{2}{d} \kappa_{d-1} \kappa_d^{-1-1/d} \Gamma\left(1 + \frac{1}{d}\right) \text{Per}(A). \end{aligned}$$

□

## 4.2 Asymptotics of higher moments

To prove Theorem 2.2 and Theorem 2.3, we need a number of lemmas. In this section  $C$  is always some constant independent of  $\lambda$  and  $A$ . Our first statement is the following version of Hölder's inequality.

**Lemma 4.2.** *For any events  $A_1, \dots, A_m$  it holds*

$$\mathbb{P} \left( \bigcap_{r=1}^m A_r \right) \leq \prod_{r=1}^m (\mathbb{P}(A_r))^{1/m}.$$

**Lemma 4.3.** *Let  $x_0, y_0 \in \mathbb{R}^d$ . For any  $\varepsilon > 0$  and  $m \in \mathbb{N}$  the following inequality holds:*

$$\int_{(\mathbb{R}^d)^m} (\mathbb{P}(x_0, x_1, \dots, x_m \in \nu_{\eta \cup \{y_0\}}(y_0)))^\varepsilon dx_1 \dots dx_m \leq e^{-\varepsilon \lambda \kappa_d \|x_0 - y_0\|^d / (m+1)} \left( \frac{m+1}{\varepsilon \lambda} \right)^m.$$

*Proof.* By Lemma 4.2, we have

$$\begin{aligned} & \int_{(\mathbb{R}^d)^m} (\mathbb{P}(x_0, x_1, \dots, x_m \in \nu_{\eta \cup \{y_0\}}(y_0)))^\varepsilon dx_1 \dots dx_m \\ & \leq (\mathbb{P}(x_0 \in \nu_{\eta \cup \{y_0\}}(y_0)))^{\varepsilon/(m+1)} \int_{(\mathbb{R}^d)^m} \prod_{i=1}^m (\mathbb{P}(x_i \in \nu_{\eta \cup \{y_0\}}(y_0)))^{\varepsilon/(m+1)} dx_1 \dots dx_m \\ & = e^{-\varepsilon \lambda \kappa_d \|x_0 - y_0\|^d / (m+1)} \left[ \int_{\mathbb{R}^d} e^{-\varepsilon \lambda \kappa_d \|x - y_0\|^d / (m+1)} dx \right]^m. \end{aligned}$$

Using (12) completes the proof.  $\square$

**Lemma 4.4.** *For any  $a > 0$*

$$\int_{\mathbb{R}^d} \int_A e^{-a\lambda \|y-x\|^d} dy dx = \frac{\kappa_d \text{Vol}(A)}{a\lambda} \quad (17)$$

and

$$\int_{\mathbb{R}^d \setminus A} \int_A e^{-a\lambda \|y-x\|^d} dy dx \leq C \frac{\text{Per}(A)}{\lambda^{1+1/d}}, \quad \lambda \rightarrow \infty. \quad (18)$$

*Proof.* The first equation follows from (10) after replacing  $\lambda$  by  $\lambda' a / \kappa_d$ . The second estimate follows from (11) after replacing  $\lambda$  by  $\lambda' a / \kappa_d$  and then applying Theorem 2.1.  $\square$

Introduce the notation  $B_r^x$  for the closed ball with center  $x \in \mathbb{R}^d$  and radius  $r > 0$  in Euclidean metric.

**Lemma 4.5.** *Let  $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ . If  $B_{\|x_1-y_1\|}^{x_1} \cap B_{\|x_2-y_2\|}^{x_2} \neq \emptyset$ , then*

$$\mathbb{P} \left( B_{\|x_1-y_1\|}^{x_1} \cap \eta = \emptyset, B_{\|x_2-y_2\|}^{x_2} \cap \eta = \emptyset \right) \leq 2 \exp \left( -\frac{\lambda \kappa_d}{2^{2d+1}} \left( \|x_1 - y_2\|^d + \|x_2 - y_1\|^d \right) \right).$$

*Proof.* Since  $B_{\|x_1-y_1\|}^{x_1} \cap B_{\|x_2-y_2\|}^{x_2} \neq \emptyset$ , it follows from the triangle inequality that

$$\frac{\|x_1 - y_2\|}{4}, \frac{\|x_2 - y_1\|}{4} \leq \max(\|x_1 - y_1\|, \|x_2 - y_2\|).$$

Therefore, by Lemma 4.2 and stationarity of  $\eta$  we have

$$\begin{aligned} & \mathbb{P} \left( B_{\|x_1-y_1\|}^{x_1} \cap \eta = \emptyset, B_{\|x_2-y_2\|}^{x_2} \cap \eta = \emptyset \right) \\ & \leq \mathbb{P} \left( B_{\|x_1-y_2\|/4}^{x_1} \cap \eta = \emptyset, B_{\|x_2-y_1\|/4}^{x_1} \cap \eta = \emptyset \text{ or } B_{\|x_1-y_2\|/4}^{x_2} \cap \eta = \emptyset, B_{\|x_2-y_1\|/4}^{x_2} \cap \eta = \emptyset \right) \\ & \leq \sum_{i=1}^2 \mathbb{P} \left( B_{\|x_1-y_2\|/4}^{x_i} \cap \eta = \emptyset, B_{\|x_2-y_1\|/4}^{x_i} \cap \eta = \emptyset \right) \\ & \leq \sum_{i=1}^2 \left( \mathbb{P} \left( B_{\|x_1-y_2\|/4}^{x_i} \cap \eta = \emptyset \right) \mathbb{P} \left( B_{\|x_2-y_1\|/4}^{x_i} \cap \eta = \emptyset \right) \right)^{1/2} \\ & = 2 \exp \left( -\frac{\lambda \kappa_d}{2^{2d+1}} \left( \|x_1 - y_2\|^d + \|x_2 - y_1\|^d \right) \right). \end{aligned}$$

□

**Lemma 4.6.** *For any  $x_1, y_1, \dots, x_n, y_n \in \mathbb{R}^d$  it holds*

$$\begin{aligned} & \mathbb{P} \left( B_{\|x_r-y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n \right) \leq \exp \left( -\lambda \kappa_d \sum_{r=1}^n \|x_r - y_r\|^d \right) \\ & + 2 \sum_{s < t} \exp \left( -\frac{\lambda \kappa_d}{n+1} \sum_{r=1}^n \|x_r - y_r\|^d \right) \exp \left( -\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} \left( \|x_s - y_t\|^d + \|x_t - y_s\|^d \right) \right). \end{aligned}$$

*Proof.* If the balls  $B_{\|x_r-y_r\|}^{x_r}$ ,  $r = 1, \dots, n$  are pairwise disjoint then we obviously have

$$\mathbb{P} \left( B_{\|x_r-y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n \right) = \exp \left( -\lambda \kappa_d \sum_{r=1}^n \|x_r - y_r\|^d \right).$$

Suppose that for some indices  $s \neq t$  it holds  $B_{\|x_s - y_s\|}^{x_s} \cap B_{\|x_t - y_t\|}^{x_t} \neq \emptyset$ . Applying Lemma 4.2, we get

$$\begin{aligned} & \mathbb{P} \left( B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n \right) \\ & \leq \left( \mathbb{P} \left( B_{\|x_s - y_s\|}^{x_s} \cap \eta = \emptyset, B_{\|x_t - y_t\|}^{x_t} \cap \eta = \emptyset \right) \right)^{1/(n+1)} \prod_{r=1}^n \left( \mathbb{P} \left( B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset \right) \right)^{1/(n+1)} \\ & = \exp \left( -\frac{\lambda \kappa_d}{n+1} \sum_{r=1}^n \|x_r - y_r\|^d \right) \left( \mathbb{P} \left( B_{\|x_s - y_s\|}^{x_s} \cap \eta = \emptyset, B_{\|x_t - y_t\|}^{x_t} \cap \eta = \emptyset \right) \right)^{1/(n+1)}. \end{aligned}$$

It remains to apply Lemma 4.5 to finish the proof.  $\square$

*Proof of Theorem 2.2.* We have

$$\begin{aligned} \mathbb{E} \text{Vol}^n(A_\eta) &= \mathbb{E} \int_{(\mathbb{R}^d)^n} \mathbf{1}(\exists(y_1, \dots, y_n) \in (\eta \cap A)^n : x_i \in \nu_\eta(y_i), i = 1, \dots, n) dx_1 \dots dx_n \\ &= \sum_{i=1}^n \sum_{m_1 + \dots + m_i = n} B_{n,i,m_1, \dots, m_i} \beta_{i,m_1, \dots, m_i}, \quad (19) \end{aligned}$$

where

$$\beta_{i,m_1, \dots, m_i} = \int_{(\mathbb{R}^d)^n} \mathbb{E} \sum_{(y_1, \dots, y_i) \in (\eta \cap A)^i} \mathbf{1}(x_1, \dots, x_{m_1} \in \nu_\eta(y_1), \dots, x_{n-m_i+1}, \dots, x_n \in \nu_\eta(y_i)) dx_1 \dots dx_n$$

and  $B_{n,i,m_1, \dots, m_i}$  denotes the number of ways to divide the set  $\{1, 2, \dots, n\}$  into  $i$  subsets of size  $m_1, \dots, m_i$ . It is clear that

$$B_{n,n,1, \dots, 1} = 1. \quad (20)$$

Fix some  $i$  and  $m_1, \dots, m_i$ . Using the Slivnyak-Mecke formula (9) we get

$$\beta_{i,m_1, \dots, m_i} = \lambda^i \int_{(\mathbb{R}^d)^n} \int_{A^i} \mathbb{P}(x_1, \dots, x_{m_1} \in \nu_{\eta \cup \tilde{y}_i}(y_1), \dots, x_{n-m_i+1}, \dots, x_n \in \nu_{\eta \cup \tilde{y}_i}(y_i)) dy_1 \dots dy_i dx_1 \dots dx_n,$$

where  $\tilde{y}_i = \{y_1, \dots, y_i\}$ . Taking into account that  $v_{\eta \cup \tilde{y}_i}(y_r) \subset v_{\eta \cup \{y_r\}}(y_r)$ , and using Fubini's theorem, Lemma 4.2 and Lemma 4.3 we obtain

$$\begin{aligned} \beta_{i,m_1,\dots,m_i} &\leq \lambda^i \int_{A^i} \prod_{r=1}^i \int_{(\mathbb{R}^d)^{m_r}} (\mathbb{P}(x_1, \dots, x_{m_r} \in v_{\eta \cup \{y_r\}}(y_r)))^{1/i} dx_1 \dots dx_{m_r} dy_1 \dots dy_i \\ &\leq \lambda^i \int_{A^i} \prod_{r=1}^i \left(\frac{im_r}{\lambda}\right)^{m_r-1} \int_{\mathbb{R}^d} \left(e^{-\frac{1}{i}\lambda\kappa_d\|x_1-y_r\|^d/m_r}\right) dx_1 dy_1 \dots dy_i. \end{aligned}$$

By (17) we get

$$\beta_{i,m_1,\dots,m_i} \leq C \text{Vol}^i(A) \lambda^{i-\sum_{r=1}^i m_r} = C \text{Vol}^i(A) \lambda^{i-n}.$$

The maximum order of  $\lambda$  is achieved for  $i = n$ , which together with (19) and (20) implies

$$\begin{aligned} \mathbb{E} \text{Vol}^n(A_\eta) &\leq \lambda^n \int_{(\mathbb{R}^d)^n} \int_{A^n} \mathbb{P}(x_r \in v_{\eta \cup \tilde{y}_n}(y_r), r = 1, \dots, n) dy_1 \dots dy_n dx_1 \dots dx_n \\ &\quad + C(\text{Vol}(A))^{n-1} \lambda^{-1}, \quad \lambda \rightarrow \infty. \end{aligned}$$

It is clear that

$$\mathbb{P}(x_r \in v_{\eta \cup \tilde{y}_n}(y_r), r = 1, \dots, n) \leq \mathbb{P}\left(B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n\right).$$

Therefore, by Lemma 4.6,

$$\mathbb{E} \text{Vol}^n(A_\eta) \leq v_n + 2 \sum_{s < t} v_{n,s,t} + C(\text{Vol}(A))^{n-1} \lambda^{-1}, \quad \lambda \rightarrow \infty, \quad (21)$$

where

$$v_n = \lambda^n \int_{(\mathbb{R}^d)^n} \int_{A^n} \exp\left(-\lambda\kappa_d \sum_{r=1}^n \|x_r - y_r\|^d\right) dy_1 \dots dy_n dx_1 \dots dx_n,$$

and

$$\begin{aligned} v_{n,s,t} &= \lambda^n \int_{(\mathbb{R}^d)^n} \int_{A^n} \exp\left(-\frac{\lambda\kappa_d}{n+1} \sum_{r=1}^n \|x_r - y_r\|^d\right) \\ &\quad \times \exp\left(-\frac{\lambda\kappa_d}{2^{2d+1}(n+1)} \left(\|x_s - y_t\|^d + \|x_t - y_s\|^d\right)\right) dy_1 \dots dy_n dx_1 \dots dx_n. \end{aligned}$$

By formula (10),

$$v_n = \text{Vol}^n(A). \quad (22)$$

Let us estimate  $v_{n,s,t}$ . Using Fubini, it follows from (17) that

$$\begin{aligned} v_{n,s,t} &\leq C \text{Vol}^{n-2}(A) \lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_A \int_A \exp\left(-\frac{\lambda \kappa_d}{(n+1)} (\|x_s - y_s\|^d + \|x_t - y_t\|^d)\right) \\ &\quad \times \exp\left(-\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) dy_t dy_s dx_t dx_s \\ &\leq C \text{Vol}^{n-2}(A) \lambda^2 \int_{\mathbb{R}^d} \int_A \exp\left(-\frac{\lambda \kappa_d}{(n+1)} (\|x_s - y_s\|^d)\right) \\ &\quad \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(-\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) dy_t dx_t dy_s dx_s. \end{aligned}$$

Furthermore, by (12),

$$v_{n,s,t} \leq C \text{Vol}^{n-2}(A) \int_{\mathbb{R}^d} \int_A \exp\left(-\frac{\lambda \kappa_d}{(n+1)} \|x_s - y_s\|^d\right) dy_s dx_s,$$

and applying (17) again, we get

$$v_{n,s,t} \leq C \text{Vol}^{n-1}(A) \lambda^{-1}.$$

Combining this with the estimate (21) and with (22), we get

$$\mathbb{E} \text{Vol}^n(A_\eta) \leq \text{Vol}^n(A) + C \text{Vol}^{n-1}(A) \lambda^{-1}, \quad \lambda \rightarrow \infty.$$

The application of Lyapunov's inequality

$$\mathbb{E} \text{Vol}^n(A_\eta) \geq (\mathbb{E} \text{Vol}(A_\eta))^n = \text{Vol}^n(A)$$

finishes the proof. □

*Proof of Theorem 2.3.* We have

$$\mathbb{E} \text{Vol}^n(A \Delta A_\eta) = \mathbb{E} (\text{Vol}(A \setminus A_\eta) + \text{Vol}(A_\eta \setminus A))^n = \sum_{k=0}^n \binom{n}{k} u_k, \quad (23)$$

where

$$u_k = \mathbb{E} \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \mathbf{1}(x_1, \dots, x_k \in A_\eta, x_{k+1}, \dots, x_n \notin A_\eta) dx_1 \dots dx_n. \quad (24)$$

Fix some  $k$ . We have

$$\begin{aligned}
u_k &= \mathbb{E} \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \mathbf{1} \left( \exists (y_1, \dots, y_k) \in (\eta \cap A)^k, (y_{k+1}, \dots, y_n) \in (\eta \setminus A)^{n-k} : \right. \\
&\qquad \qquad \qquad \left. x_i \in v_\eta(y_i), i = 1, \dots, n \right) dx_1 \dots dx_n \\
&= \sum_{i=1}^k \sum_{j=1}^{n-k} \sum_{m_1 + \dots + m_i = k} \sum_{l_1 + \dots + l_j = n-k} B_{k,i,m_1, \dots, m_i} B_{n-k,j,l_1, \dots, l_j} \beta_{i,j,m_1, \dots, m_i, l_1, \dots, l_j}, \quad (25)
\end{aligned}$$

where

$$\begin{aligned}
\beta_{i,j,m_1, \dots, m_i, l_1, \dots, l_j} &= \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \mathbb{E} \sum_{(y_1, \dots, y_i) \in (\eta \cap A)^i} \sum_{(y_{i+1}, \dots, y_{i+j}) \in (\eta \setminus A)^j} \\
&\qquad \qquad \qquad \mathbf{1} (x_1, \dots, x_{m_1} \in v_\eta(y_1), \dots, x_{n-l_j+1}, \dots, x_n \in v_\eta(y_{i+j})) \\
&\qquad \qquad \qquad dx_1 \dots dx_n
\end{aligned}$$

and  $B_{k,i,m_1, \dots, m_i}, B_{n-k,j,l_1, \dots, l_j}$  are the same combinatorial coefficients as in the proof of Theorem 2.2.

Fix some  $i, j$ , and  $m_1, \dots, m_i, l_1, \dots, l_j$ . Using the Slivnyak-Mecke formula (9) twice we get

$$\begin{aligned}
\beta_{i,j,m_1, \dots, m_i, l_1, \dots, l_j} &= \lambda^{i+j} \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \int_{(\mathbb{R}^d \setminus A)^j} \int_{A^i} \\
&\qquad \qquad \qquad \mathbb{P} (x_1, \dots, x_{m_1} \in v_{\eta \cup \tilde{y}_{i+j}}(y_1), \dots, x_{n-l_j+1}, \dots, x_n \in v_{\eta \cup \tilde{y}_{i+j}}(y_{i+j})) \\
&\qquad \qquad \qquad dy_1 \dots dy_{i+j} dx_1 \dots dx_n,
\end{aligned}$$

where  $\tilde{y}_{i+j} = \{y_1, \dots, y_{i+j}\}$ . By Fubini and Lemma 4.2,

$$\begin{aligned}
\beta_{i,j,m_1, \dots, m_i, l_1, \dots, l_j} &\leq \lambda^{i+j} \int_{(\mathbb{R}^d \setminus A)^j} \int_{A^i} \\
&\qquad \qquad \qquad \prod_{r=1}^i \int_{(\mathbb{R}^d \setminus A)^{m_r}} (\mathbb{P} (x_1, \dots, x_{m_r} \in v_{\eta \cup \{y_r\}}(y_r)))^{1/(i+j)} dx_1 \dots dx_{m_r} \\
&\qquad \qquad \qquad \times \prod_{r=1}^j \int_{A^{l_r}} (\mathbb{P} (x_1, \dots, x_{l_r} \in v_{\eta \cup \{y_{i+r}\}}(y_{i+r})))^{1/(i+j)} dx_1 \dots dx_{l_r} \\
&\qquad \qquad \qquad dy_1 \dots dy_{i+j}.
\end{aligned}$$

Using Lemma 4.3 and (18), we get asymptotically as  $\lambda \rightarrow \infty$

$$\begin{aligned} \beta_{i,j,m_1,\dots,m_i,l_1,\dots,l_j} &\leq C \operatorname{Per}(A)^{i+j} \lambda^{i+j+\sum_{r=1}^i(-m_r-1/d)+\sum_{r=1}^j(-l_r-1/d)} \\ &= C \operatorname{Per}(A)^{i+j} \lambda^{-n+i+j-(i+j)/d}. \end{aligned}$$

The maximum order of  $\lambda$  is achieved for  $i = k$ ,  $j = n - k$ , and the next term of maximum order is achieved for  $i + j = n - 1$ , which together with (25) and (20) implies

$$\begin{aligned} u_k &\leq \lambda^n \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \int_{(\mathbb{R}^d \setminus A)^{n-k}} \int_{A^k} \mathbb{P}(x_r \in v_{\eta \cup \tilde{y}_n}(y_r), r = 1, \dots, n) dy_1 \dots dy_n dx_1 \dots dx_n \\ &\quad + C \operatorname{Per}(A)^{n-1} \lambda^{-1-(n-1)/d} \end{aligned}$$

asymptotically as  $\lambda \rightarrow \infty$ . It is clear that

$$\mathbb{P}(x_r \in v_{\eta \cup \tilde{y}_n}(y_r), r = 1, \dots, n) \leq \mathbb{P}\left(B_{\|x_r - y_r\|}^{x_r} \cap \eta = \emptyset, r = 1, \dots, n\right).$$

Therefore, by Lemma 4.6, asymptotically as  $\lambda \rightarrow \infty$ ,

$$u_k \leq v_k + 2 \sum_{s < t} v_{k,s,t} + C \operatorname{Per}(A)^{n-1} \lambda^{-1-(n-1)/d}, \quad (26)$$

where

$$v_k = \lambda^n \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \int_{(\mathbb{R}^d \setminus A)^{n-k}} \int_{A^k} \exp\left(-\lambda \kappa_d \sum_{r=1}^n \|x_r - y_r\|^d\right) dy_1 \dots dy_n dx_1 \dots dx_n,$$

and

$$\begin{aligned} v_{k,s,t} &= \lambda^n \int_{A^{n-k}} \int_{(\mathbb{R}^d \setminus A)^k} \int_{(\mathbb{R}^d \setminus A)^{n-k}} \int_{A^k} \exp\left(-\frac{\lambda \kappa_d}{n+1} \sum_{r=1}^n \|x_r - y_r\|^d\right) \\ &\quad \times \exp\left(-\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) dy_1 \dots dy_n dx_1 \dots dx_n. \end{aligned}$$

By the identity (11),

$$v_k = 2^{-n} (\mathbb{E} \operatorname{Vol}(A \Delta A_\eta))^n. \quad (27)$$

Let us estimate  $v_{k,s,t}$ . For instance, we assume that  $s \leq k$  and  $t \geq k + 1$  (other cases are treated in the same way). In the same way as in the proof of Theorem 2.2, we obtain by

inequality (18)

$$v_{k,s,t} \leq C \operatorname{Per}(A)^{n-2} \lambda^{2-(n-2)/d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus A} \int_{\mathbb{R}^d} \int_A \exp\left(-\frac{\lambda \kappa_d}{(n+1)} (\|x_s - y_s\|^d)\right) \\ \times \exp\left(-\frac{\lambda \kappa_d}{2^{2d+1}(n+1)} (\|x_s - y_t\|^d + \|x_t - y_s\|^d)\right) dy_s dy_t dx_s dx_t$$

as  $\lambda \rightarrow \infty$ . Furthermore, by (12),

$$v_{k,s,t} \leq C \operatorname{Per}(A)^{n-2} \lambda^{-(n-2)/d} \int_{\mathbb{R}^d \setminus A} \int_A \exp\left(-\frac{\lambda \kappa_d}{(n+1)} \|x_s - y_s\|^d\right) dy_s dx_s, \quad \lambda \rightarrow \infty,$$

and applying (18) again, we get

$$v_{k,s,t} \leq C \operatorname{Per}(A)^{n-1} \lambda^{-1-(n-1)/d}, \quad \lambda \rightarrow \infty.$$

Combining this with (26) and (27), we get

$$u_k \leq 2^{-n} (\mathbb{E} \operatorname{Vol}(A \Delta A_\eta))^n + C \operatorname{Per}(A)^{n-1} \lambda^{-1-(n-1)/d}, \quad \lambda \rightarrow \infty. \quad (28)$$

Inserting this into (23) we obtain

$$\mathbb{E} \operatorname{Vol}^n(A \Delta A_\eta) \leq (\mathbb{E} \operatorname{Vol}(A \Delta A_\eta))^n + C \operatorname{Per}(A)^{n-1} \lambda^{-1-(n-1)/d}, \quad \lambda \rightarrow \infty.$$

The application of Lyapunov's inequality

$$\mathbb{E} \operatorname{Vol}^n(A \Delta A_\eta) \geq (\mathbb{E} \operatorname{Vol}(A \Delta A_\eta))^n$$

finishes the proof.  $\square$

*Proof of Corollary 2.2.* As was mentioned above, the second inequality immediately follows from Theorem 2.3. To prove the first one, let us again combine (28) and (23) now for  $n = 2$  and  $k = 0, 1, 2$ . We get for sufficiently large  $\lambda$

$$\mathbb{E} \operatorname{Vol}^2(A \setminus A_\eta) + \mathbb{E} \operatorname{Vol}^2(A_\eta \setminus A) \leq \frac{1}{2} (\mathbb{E} \operatorname{Vol}(A \Delta A_\eta))^2 + 2C \operatorname{Per}(A) \lambda^{-1-1/d},$$

$$2\mathbb{E} (\operatorname{Vol}(A \setminus A_\eta) \operatorname{Vol}(A_\eta \setminus A)) \leq \frac{1}{2} (\mathbb{E} \operatorname{Vol}(A \Delta A_\eta))^2 + 2C \operatorname{Per}(A) \lambda^{-1-1/d}.$$

Combining this with Lyapunov's inequality

$$\mathbb{E} \operatorname{Vol}^2(A \setminus A_\eta) + \mathbb{E} \operatorname{Vol}^2(A_\eta \setminus A) + 2\mathbb{E} (\operatorname{Vol}(A \setminus A_\eta) \operatorname{Vol}(A_\eta \setminus A)) \geq (\mathbb{E} \operatorname{Vol}(A \Delta A_\eta))^2,$$

we obtain for sufficiently large  $\lambda$

$$\mathbb{E} \operatorname{Vol}^2(A \setminus A_\eta) + \mathbb{E} \operatorname{Vol}^2(A_\eta \setminus A) - 2\mathbb{E} (\operatorname{Vol}(A \setminus A_\eta) \operatorname{Vol}(A_\eta \setminus A)) \leq 4C \operatorname{Per}(A) \lambda^{-1-1/d},$$

which together with (13) completes the proof.  $\square$

## References

- [1] L. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems* (2000), Oxford University Press.
- [2] G. Averkov and G. Bianchi, *Confirmation of Matheron's conjecture on the covariogram of a planar convex body.*, J. Eur. Math. Soc. **11** (2009), 1187–1202.
- [3] P. Billingsley, *Probability and measure* (1979), J. Wiley & Sons, New York.
- [4] J. H. J. Einmahl and E. V. Khmaladze, *The two-sample problem in  $\mathbb{R}^m$  and measure-valued martingales.* In: *State of the art in probability and statistics (Leiden, 1999)*. IMS Lecture Notes Monogr. Ser. **36** (2001), 434–463.
- [5] B. Galerne, *Computation of the perimeter of measurable sets via their covariogram. Applications to random sets.*, Image Anal. Stereol., **30** (2011), 39–51.
- [6] L. Heinrich, *Asymptotic methods in statistics of random point processes*, in Spodarev, E. (ed.), *Lectures on stochastic geometry, spatial statistics and random fields: Asymptotic methods*. Lecture Notes in Mathematics (2012), Springer (to appear).
- [7] M. Heveling, M. Reitzner, *Poisson-Voronoi approximation.* Ann. Appl. Probab., **19** (2009), 719–736.
- [8] E. Khmaladze and N. Toronjadze, *On the almost sure coverage property of Voronoi tessellation: the  $\mathbb{R}^1$  case*, Adv. Appl. Probab. **33** (2001), 756–764.
- [9] G. Last and M. Penrose, *Poisson process Fock space representation, chaos expansion and covariance inequalities*, Probab. Theory Relat. Fields **150**, No. 3–4 (2011), 663–690.
- [10] J. Møller, *Lectures on random Voronoi tessellations*, Lecture Notes in Statistics **87** (1994), Springer, New York.
- [11] M. D. Penrose, *Laws of large numbers in stochastic geometry with statistical applications*, Bernoulli **13** (2007), 1124–1150.
- [12] M. Schulte, *A central limit theorem for the Poisson-Voronoi approximation*, manuscript, arXiv:1111.6466
- [13] R. Schneider and W. Weil, *Stochastic and integral geometry* (2008), Springer, Berlin.
- [14] D. Stoyan, W. S. Kendall and J. Mecke, *Stochastic geometry and its applications*, 2nd ed. (1995), Wiley, Chichester.

- [15] L. Wu, *A new modified logarithmic Sobolev inequality for Poisson point processes and several applications*, Probab. Theory Relat. Fields **118** (2000), 427–438.