SIMULATION OF INFINITELY DIVISIBLE RANDOM FIELDS

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ABSTRACT. Two methods to approximate infinitely divisible random fields are presented. The methods are based on approximating the kernel function in the spectral representation of such fields, leading to numerical integration of the respective integrals. Error bounds for the approximation error are derived and the approximations are used to simulate certain classes of infinitely divisible random fields.

1. INTRODUCTION

In many cases, the normal distribution is a reasonable model for real phenomena. If one considers the cumulative outcome of a great amount of influence factors, the normal distribution assumption can be justified by the Central Limit Theorem which states that the sum of a large number of independent and identically distributed random variables can be approximated by a normal distribution if the variance of these variables is finite. However, many real phenomena exhibit rather heavy tails. Stable distributions remedy this drawback by still being the limit distribution of a sum of independent and identically distributed random variables, but allowing for an infinite variance and heavy tails.

Stable distributions are a prominent example of the class of infinitely divisible distributions which we particularly concentrate on in this paper. Infinitely divisible distributions are distributions whose probability measure $\mathbb{P}$ is equal to the n-fold convolution of a probability measure $\mathbb{P}_n$ for any positive integer n. The class of infinitely divisible distributions comprises further well-known examples such as the Poisson, geometric, negative binomial, exponential, and gamma distribution, see [19]. These distributions are widely used in practice, for instance in finance to model the returns of stocks or in insurance to model the claim amounts and the number of claims of an insurance portfolio.

In order to include time dependencies or the spatial structure of real phenomena, random functions, in particular processes or fields, may be an appropriate model. An example of infinitely divisible processes are Lévy processes which have been extensively studied in the literature.

In this paper, we consider random fields that can be represented as a stochastic integral of a deterministic kernel function as integrand and an infinitely divisible random measure as integrator. The kernel function basically determines the dependence structure, where-as the infinitely divisible random measure inhibits the probabilistic characteristics of the random field, cf. Section 2 for details. As already

noticed in [6], practitioners have to try a variety of kernels and infinitely divisible random measures to find the model that best fits their needs.

Once the model is fixed, it is desirable to be able to perform simulations of the considered random field. There are several papers that are devoted to this problem. In [4, 21, 26], the fast Fourier transform is used for the simulation of linear fractional stable processes, whereas in [9], a wavelet representation of a certain type of fractional stable processes is applied to simulate sample paths. Furthermore, a general framework for the simulation of fractional fields is given in [6].

In this paper, we consider infinitely divisible random fields for which the kernel functions are assumed to be Hölder-continuous or bounded which is a less restrictive assumption. Based on these assumptions, we derive estimates for the approximation error when the kernel functions are approximated by step functions or by certain truncated wavelet series. The approximation allows for simulation since the integral representation of the random field reduces to a finite sum of random variables in this case.

In Section 3, we present the main results for the approximation error which is made when the kernel function is replaced by a step function or a truncated wavelet series. Section 4 is devoted to a brief simulation study where we apply the derived formulas for the approximation error to the simulation of two particular stable random fields. Finally, in Section 5 we comment on the simulation results and the methods discussed in Section 3.

2. Infinitely divisible random fields admitting an integral representation

Let Λ be an infinitely divisible random measure with control measure λ, cf. [11] and [17]. Let $f_t : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, be Λ-*integrable* for all $t \in \mathbb{R}^q$, $q \geq 1$, that is there exists a sequence of simple functions $\{\tilde{f}_t^{(n)}\}_{n \in \mathbb{N}}$, $\tilde{f}_t^{(n)} : \mathbb{R}^d \to \mathbb{R}$, $t \in \mathbb{R}^q$, so that

(a) $\tilde{f}_t^{(n)} \to f_t \quad \lambda - $a.e.,
(b) for every Borel set $B \subset \mathbb{R}^d$, the sequence $\{\int_B \tilde{f}_t^{(n)}(x)\Lambda(dx)\}_{n \in \mathbb{N}}$ converges in probability,

for all $t \in \mathbb{R}^q$. The integral of a simple function $f = \sum_{j=1}^m x_j \mathbb{1}_{A_j}$, $m \in \mathbb{N}$, is defined in the obvious manner $\int_A f d\Lambda := \sum_{j=1}^m x_j \Lambda(A \cap A_j)$. For each $t \in \mathbb{R}^q$, we set

$$\int_{\mathbb{R}^d} f_t(x)\Lambda(dx) := \plim_{n \to \infty} \int_{\mathbb{R}^d} \tilde{f}_t^{(n)}(x)\Lambda(dx),$$

cf. [17], where $\plim_{n \to \infty}$ means convergence in probability, and consider random fields of the form

$$(2.1) \qquad X(t) = \int_{\mathbb{R}^d} f_t(x)\Lambda(dx), \quad t \in \mathbb{R}^q.$$

Remark 2.1. In [17], it is shown that $X(t)$ is infinitely divisible for all $t \in \mathbb{R}^q$, cf. Theorem 2.7 and the Lévy form of the characteristic function of an infinitely divisible random variable. More generally, it can be shown that $(X(t_1), \dots, X(t_n))^\mathsf{T}$ is infinitely divisible for all $t_1, \dots, t_n \in \mathbb{R}^q$ and $n \in \mathbb{N}$ (see [14]). Therefore, any random field of the form (2.1) is an infinitely divisible random field.

Example 2.2 (α-stable random fields)**.** Let $0 < \alpha \leq 2$, $\Lambda = M$ be an (independently scattered) α-stable random measure on $\mathbb{R}^d$ with zero drift, control measure

m and skewness intensity β, see [18]. Moreover, assume that $f_t \in L^\alpha(\mathbb{R}^d, m)$ if $\alpha \neq 1$ and that f_t belongs to the set

$$\{f \in L^1(\mathbb{R}^d, m) : \int_{\mathbb{R}^d} |f(x)\beta(x)\ln|f(x)||m(dx) < \infty\}$$

if $\alpha = 1$ for all $t \in \mathbb{R}^q$. We denote the set of all functions f_t satisfying these conditions by $\mathcal{F}$. Then

$$X(t) = \int_{\mathbb{R}^d} f_t(x)M(dx), \quad t \in \mathbb{R}^q,$$

is an α-*stable random field*.

Example 2.3 (Shot noise fields)**.** Let $\Lambda = \Phi$ be a Poisson random measure on $\mathbb{R}^d$ with intensity measure Θ, see [22]. Furthermore, we assume that f_t is a measurable function on $\mathbb{R}^d$ for each $t \in \mathbb{R}^q$. Then we can consider the *shot noise field*

$$X(t) = \int_{\mathbb{R}^d} f_t(x)\Phi(dx), \quad t \in \mathbb{R}^q,$$

which can be written as

(2.2) $$X(t) = \sum_{x \in \Psi} f_t(x),$$

where Ψ is the support set of the Poisson random measure.

Example 2.4 (Lévy processes)**.** Let Q be a Poisson random measure on the set $(0, \infty) \times \mathbb{R} \setminus \{0\}$ with intensity measure $\mu \times \nu$. Here, μ is the Lebesgue measure and ν is the Lévy measure, cf. [15, 16]. Let G be a Gaussian (2-stable) independently scattered random measure with Lebesgue control measure and skewness intensity $\beta \equiv 0$. Then for any $t \geq 0$

(2.3) $$X(t) = \int_0^t \int_{\mathbb{R}} xQ(dx, ds) + t\left(\gamma - \int_{|x|<1} x\nu(dx)\right) + \int_0^t G(ds)$$

(2.4) $$= \int_{\mathbb{R}} \mathbb{1}(0 \leq s \leq t)\Lambda(ds)$$

with $\Lambda(ds) = \int_{\mathbb{R}} xQ(dx, ds) + G(ds) + \left(\gamma - \int_{|x|<1} x\nu(dx)\right)\mu(ds)$ is the *Lévy process* with Lévy measure ν, Gaussian part G and drift $\gamma = \mathbb{E}\left(X(1) - \int_{|x|\geq 1} x\nu(dx)\right)$ if $\mathbb{E}X(1)$ exists and is finite.

We now consider the cumulant function $C_{\Lambda(A)}(t) = \ln(\mathbb{E}\exp\{it\Lambda(A)\})$ of $\Lambda(A)$ for a set A in the δ-ring $\mathcal{A}$ of bounded Borel subsets of $\mathbb{R}^d$ which is given by the *Lévy-Khintchine representation*

$$C_{\Lambda(A)}(v) = iva(A) - \frac{1}{2}v^2 b(A) + \int_{\mathbb{R}} (e^{ivr} - 1 - ivr\mathbb{1}_{[-1,1]}(r))U(dr, A)$$

for $v \in \mathbb{R}$, where a is a σ-additive set function on $\mathcal{A}$, b is a measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, and $U(dr, A)$ is a measure on $\mathcal{B}(\mathbb{R}^d)$ for fixed dr and a Lévy measure on $\mathcal{B}(\mathbb{R})$ for each fixed $A \in \mathcal{B}(\mathbb{R}^d)$, that is $U(\{0\}, A) = 0$ and $\int_{\mathbb{R}} \min\{1, r^2\}U(dr, A) < \infty$, cf. [10]. The measure U is referred to as the *generalized Lévy measure* and (a, b, U) is called *characteristic triplet*. The control measure

λ can be written as

$$(2.5) \qquad \lambda(A) = |a|(A) + b(A) + \int_{\mathbb{R}} \min\{1, r^2\} U(dr, A), \quad A \in \mathcal{A},$$

where $|a| = a^+ + a^-$, see [17]. Furthermore, a and b are absolutely continuous with respect to λ and we have the formulas $a(d\eta) = \tilde{a}(\eta)\lambda(d\eta)$, $b(d\eta) = \tilde{b}(\eta)\lambda(d\eta)$ and $U(dr, d\eta) = V(dr, \eta)\lambda(d\eta)$, where $V(dr, \eta)$ is a Lévy measure for fixed η.

We now introduce the so-called *spot variable* $L'(\eta)$ with cumulant function

$$C_{L'(\eta)}(v) = iv\tilde{a}(\eta) - \frac{1}{2}v^2\tilde{b}(\eta) + \int_{\mathbb{R}} (e^{ivr} - 1 - ivr\mathbb{1}_{[-1,1]}(r))V(dr, \eta)$$

with

$$(2.6) \qquad \mathbb{E}(L'(\eta)) = \tilde{a}(\eta) + \int_{[-1,1]^C} rV(dr, \eta),$$

$$(2.7) \qquad \mathrm{Var}(L'(\eta)) = \tilde{b}(\eta) + \int_{\mathbb{R}} r^2 V(dr, \eta)$$

if $\mathbb{E}(L'(\eta))$ and $\mathrm{Var}(L'(\eta))$ exist. Here, $[-1, 1]^C$ means the complement of the interval $[-1, 1]$ in $\mathbb{R}$. It is shown in [10] that $C_{X(t)}(v) = \int_{\mathbb{R}^d} C_{L'(\eta)}(vf_t(\eta))\lambda(d\eta)$. We can use the cumulant function of $X(t)$ to obtain the second moment of $X(t)$ (in case it exists):

$$(2.8) \qquad \mathbb{E}\left(X(t)^2\right) = \int_{\mathbb{R}^d} f_t^2(y)\mathrm{Var}(L'(y))\lambda(dy) + \left(\int_{\mathbb{R}^d} f_t(y)\mathbb{E}(L'(y))\lambda(dy)\right)^2.$$

As noticed in [10], it is no restriction for modeling purposes if we only consider characteristic triplets of the form $a(d\eta) = \tilde{a}_\nu(\eta)\nu(d\eta)$, $b(d\eta) = \tilde{b}_\nu(\eta)\nu(d\eta)$ and $U(dr, d\eta) = V_\nu(dr, d\eta)\nu(d\eta)$, where ν is a non-negative measure on $\mathcal{B}(\mathbb{R}^d)$, the functions $\tilde{a}_\nu : \mathbb{R}^d \to \mathbb{R}$ and $\tilde{b}_\nu : \mathbb{R}^d \to [0, \infty)$ are measurable and $V_\nu(dr, \eta)$ is a Lévy measure for fixed η.

Example 2.5. The following choice of (a, b, U) will lead to the so-called *gamma Lévy basis* in Example 3.4. We choose the Lebesgue measure for ν and consider the characteristic triplet $(a, 0, U)$ with $U(dr, d\eta) = V(dr, \eta)d\eta = \mathbb{1}_{(0,\infty)}(r)\frac{1}{r}e^{-\theta r}drd\eta$ and $a(d\eta) = \tilde{a}(\eta)d\eta = \frac{1}{\theta}\left(1 - e^{-\theta}\right)d\eta$, where $\theta \in (0, \infty)$. Then, by using (2.6) and (2.7), we get $\mathbb{E}(L'(\eta)) = 1/\theta$ and $\mathrm{Var}(L'(\eta)) = 1/\theta^2$ and by (2.5), the control measure is proportional to the Lebesgue measure with

$$\lambda(d\eta) = \left(\frac{1 + \theta - 2\theta e^{-\theta} - e^{-\theta}}{\theta^2} + \int_1^\infty \frac{1}{r}e^{-\theta r}dr\right)d\eta.$$

3. Approximation of infinitely divisible random fields

We now restrict our setting to the observation window $[-T, T]^q$ with $T > 0$ so that $X(t) = \int_{\mathbb{R}^d} f_t(x)\Lambda(dx)$ for $t \in [-T, T]^q$. We denote by $supp(f_t)$ the support of f_t for each $t \in [-T, T]^q$ and assume that $\bigcup_{t \in [-T,T]^q} supp(f_t) \subset [-A, A]^d$ for an $A > 0$. Then $X(\cdot)$ can be written as

$$X(t) = \int_{[-A,A]^d} f_t(x)\Lambda(dx), \quad t \in [-T, T]^q.$$

Our goal is to approximate sample paths of X for a variety of kernel functions f_t, $t \in [-T, T]^q$. The idea is to approximate the kernel functions f_t appropriately so

that the approximations $\tilde{f}_t^{(n)}$ are of the form $\tilde{f}_t^{(n)} = \sum_{i=1}^{m(n)} a_i g_{t,i}$ for $t \in [-T, T]^q$, where $m(n) \in \mathbb{N}$, $a_i \in \mathbb{R}$ and $g_{t,i} : \mathbb{R}^d \to \mathbb{R}$ is Λ-integrable.

Due to the linearity of the stochastic integral, we get for each $t \in [-T, T]^q$

$$(3.1) \qquad \tilde{X}^{(n)}(t) = \int_{[-A,A]^d} \tilde{f}_t^{(n)}(x)\Lambda(dx) = \sum_{i=1}^{m(n)} a_i \int_{[-A,A]^d} g_{t,i}(x)\Lambda(dx)$$

as an approximation $\tilde{X}^{(n)}$ of X. If the functions $g_{t,i}$ are simple functions such that $\int_{[-A,A]^d} g_{t,i}(x)\Lambda(dx) = \sum_{j=1}^{l} g_{t,i}(x_j)\Lambda(\Delta_j)$ for $i = 1, \ldots, m(n)$, $t \in [-T, T]^q$, some $x_j \in [-A, A]^d$, $l \in \mathbb{N}$ and a partition $\{\Delta_j\}_{j=1}^{l}$ of $[-A, A]^d$ into pairwise disjoint Borel sets, then

$$(3.2) \qquad \tilde{X}^{(n)}(t) = \sum_{i=1}^{m(n)} \sum_{j=1}^{l} a_i g_{t,i}(x_j)\Lambda(\Delta_j)$$

which can be simulated if $\Lambda(\Delta_j)$, $j = 1, \ldots, l$, can be simulated.

Example 3.1 (α-stable random fields)**.** Let $\Lambda = M$ be an α-stable random measure with Lebesgue control measure and constant skewness intensity β. Then

$$M(\Delta_j) \sim S_\alpha(|\Delta_j|^{1/\alpha}, \beta, 0), \quad j = 1, \ldots, l,$$

cf. [18], where $|\Delta_j|$ is the volume of Δ_j and $S_\alpha(\sigma, \beta, 0)$ denotes the stable distribution with stable index α, scale parameter σ, skewness parameter β and location parameter 0. Furthermore, the random variables $M(\Delta_j)$, $j = 1, \ldots, l$, are independent since M is an independently scattered random measure. A method to simulate α-stable random variables is presented in [5].

Example 3.2 (Shot noise random fields)**.** Let $\Lambda = \Phi$ be a Poisson random measure with intensity measure Θ. Then

$$\Lambda(\Delta_j) \sim Poi(\Theta(\Delta_j)),$$

where $Poi(\Theta(\Delta_j))$ stands for the Poisson distribution with mean $\Theta(\Delta_j)$. We notice that simulating sample paths of a shot noise field X by $\tilde{X}^{(n)}$ in (3.1) is not efficient since one can directly exploit the structure of X and use $X(t) = \sum_{x \in \Psi} f_t(x)$, cf. equation (2.2) in Example 2.3. Realizations of a homogeneous Poisson process Ψ (if Θ is proportional to the Lebesgue measure) can be obtained e. g. by means of the radial simulation method, whereas for non-homoge-neous Poisson processes, the thinning method can be used, cf. [22].

Example 3.3 (Lévy processes)**.** Let Q be a Poisson random measure on the set $(0, \infty) \times \mathbb{R} \setminus \{0\}$ with intensity measure $\mu \times \nu$, where μ is the Lebesgue measure and ν is the Lévy measure. Let G be a Gaussian white noise measure with Lebesgue control measure and skewness intensity $\beta \equiv 0$. In contrast to the representation (2.4), we first approximate the Lévy process (2.3) by

$$X_K(t) = \int_{-K}^{K} \int_{0}^{t} x Q(ds, dx) - t \int_{|x|<1} x \nu(dx) + \gamma t + G([0, t])$$

for some $K > 0$. We approximate $f(x) = x$ by a linear combination of simple functions g_i, $i = 1, \ldots, m(n)$, $m(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$ and get

$$\tilde{X}_K^{(n)}(t) = \sum_{i=1}^{m(n)} \sum_{j=1}^{l} a_i g_i(x_j) Q(\Delta_j) - t \int\limits_{|x|<1} x\nu(dx) + \gamma t + G([0,t]),$$

for a partition $\{\Delta_j\}_{j=1}^l$, $l \in \mathbb{N}$, of $[-K, K] \times [0, t]$ and some $a_1, \ldots, a_{m(n)} \in \mathbb{R}$, where $Q(\Delta_j) \sim Poi((\mu \times \nu)(\Delta_j))$ and $G([0,t]) \sim \mathcal{N}(0, t)$.

Example 3.4 (Gamma Lévy random fields). Let us again choose the Lebesgue measure for ν and the characteristic triplet $(a, 0, U)$ from Example 2.5. Then $\Lambda(\Delta_j) \sim \Gamma(|\Delta_j|, \theta)$, cf. [10], where $|\Delta_j|$ is the Lebesgue measure of Δ_j and $\Gamma(|\Delta_j|, \theta)$ is the gamma distribution with probability density function

$$f(x) = \frac{\theta^{|\Delta_j|}}{\Gamma(|\Delta_j|)} x^{|\Delta_j|-1} e^{-\theta x} \mathbb{1}_{[0,\infty)}(x).$$

In the last formula, $\Gamma(\cdot)$ denotes the gamma function. Again, $\Lambda(\Delta_j)$, $j = 1, \ldots, n$, are independent. Due to its distributional property, Λ is called *gamma Lévy basis* and

$$X(t) = \int_{\mathbb{R}^d} f_t(x) \Lambda(dx), \quad t \in \mathbb{R}^q,$$

gamma Lévy random field. Since gamma distributed random variables can be easily simulated (cf. [12]), the procedure (3.2) applies immediately.

3.1. **Measuring the approximation error.** Approximating the random field X with $\tilde{X}^{(n)}$ by taking an approximation $\tilde{f}_t^{(n)}$ of the kernel functions f_t implies that $\tilde{X}^{(n)}$ is close to X when $\tilde{f}_t^{(n)}$ is close to f_t. We use

$$Err_s(X(t), \tilde{X}^{(n)}(t)) := \left\| f_t - \tilde{f}_t^{(n)} \right\|_{L^s} := \left(\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^s \lambda(dx) \right)^{1/s}$$

to measure the approximation quality of $\tilde{X}^{(n)}$ for some $s > 0$. In order to guarantee the finiteness of the above integral, we assume from now on that f_t and $\tilde{f}_t^{(n)}$ belong to $L^s([-A, A]^d, \lambda)$ for all $t \in [-T, T]^q$. The goal is then to find a set of functions $\{\tilde{f}_t^{(n)}\}_{t \in \mathbb{R}^d}$ so that $Err_s(X(t), \tilde{X}^{(n)}(t))$ is less than a predetermined critical value.

We see that the problem of approximating the random field X reduces to an approximation problem of the corresponding kernel functions.

Let us now consider two special cases where Λ is an α-stable random measure and a Poisson random measure, respectively, to analyze the choice of the error measure.

3.1.1. α-*stable random measures.* Assume that $0 < \alpha \leq 2$, $\alpha \neq 1$, and let M be an α-stable random measure with control measure m and skewness intensity β. Furthermore, if $\alpha = 1$, assume additionally that $\beta(t) = 0$ for all $t \in [-T, T]^q$. Consider a set of functions $\{\tilde{f}_t^{(n)}\}_{t \in \mathbb{R}^d}$, where $\tilde{f}_t^{(n)} \in \mathcal{F}$ (cf. Example 2.2) for all $t \in [-T, T]^q$ and $n \in \mathbb{N}$. The corresponding α-stable random field is denoted by

$$\tilde{X}^{(n)}(t) := \int_{[-A,A]^d} \tilde{f}_t^{(n)}(x) M(dx), \quad t \in [-T, T]^q.$$

We know that $\tilde{X}^{(n)}(t)$ converges stochastically to $X(t)$ in probability if and only if the quantity $\int_{\mathbb{R}^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha m(dx)$ converges to 0 as n goes to infinity (see [18]). Therefore, we can use $\tilde{X}^{(n)}(t)$ as an approximation for $X(t)$ if $\tilde{f}_t^{(n)}$ approximates f_t sufficiently well, i. e. $\tilde{X}^{(n)}(t)$ converges to $X(t)$ in probability if and only if $Err_\alpha(X(t), \tilde{X}^{(n)}(t)) \to 0$ as $n \to \infty$.

The choice of $Err_\alpha(X(t), \tilde{X}^{(n)}(t))$ can be further justified as follows. Since the random variables $X(t)$ and $\tilde{X}^{(n)}(t)$ are jointly α-stable distributed for each $t \in [-T, T]^q$, the difference $X(t) - \tilde{X}^{(n)}(t)$ is also an α-stable random variable. The scale parameter of $X(t) - \tilde{X}^{(n)}(t)$ is given by

$$\sigma_{X(t)-\tilde{X}^{(n)}(t)} = \left(\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha m(dx) \right)^{1/\alpha},$$

cf. [18], so that $Err_\alpha(X(t), \tilde{X}^{(n)}(t)) = \sigma_{X(t)-\tilde{X}^{(n)}(t)}$. Furthermore, let us consider the quantity $\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p$ for $0 < p < \alpha$, that is the mean error between $X(t)$ and $\tilde{X}^{(n)}(t)$ in the L^p-sense. Since $X(t) - \tilde{X}^{(n)}(t)$ is an α-stable random variable, we have $\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p < \infty$ if $0 < p < \alpha$ and $\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p = \infty$ if $p \geq \alpha$. For $0 < p < \alpha$, $0 < \alpha < 2$ and $\alpha \neq 1$, this quantity can be written as

$$(3.3) \qquad \left(\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p \right)^{1/p} = c_{\alpha,\beta_t}(p) \cdot \sigma_{X(t)-\tilde{X}^{(n)}(t)},$$

where

$$(c_{\alpha,\beta_t}(p))^p = \frac{2^{p-1} \Gamma(1 - \frac{p}{\alpha}) \left(1 + \beta_t^2 \tan^2 \frac{\alpha\pi}{2}\right)^{\frac{p}{2\alpha}}}{p \int_0^\infty u^{-p-1} \sin^2 u \, du} \cos\left(\frac{p}{\alpha} \arctan\left(\beta_t \tan \frac{\alpha\pi}{2} \right) \right),$$

$$\beta_t = \frac{\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha \operatorname{sign}(f_t(x) - \tilde{f}_t^{(n)}(x)) \beta(x) dx}{\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha dx}.$$

In the case $\alpha = 1$, equation (3.3) holds if $\beta_t = 0$, see [18].

We remind that $\beta(\cdot)$ is the skewness intensity of the α-stable random measure M. The above implies that for $0 < p < \alpha$

$$Err_\alpha(X(t), \tilde{X}^{(n)}(t)) = \frac{1}{c_{\alpha,\beta_t}(p)} \left(\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p \right)^{\frac{1}{p}}.$$

Now assume that $\alpha = 1$ and $\beta(x) \neq 0$ for at least one $x \in [-A, A]^d$. In this case, we need to impose an additional condition on the kernel functions in order to guarantee the convergence of $\tilde{X}^{(n)}(t)$ to $X(t)$ in probability. Namely, $\tilde{X}^{(n)}(t)$ converges to $X(t)$ in probability if and only if $\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)| m(dx)$ and $\int_{[-A,A]^d} (f_t(x) - \tilde{f}_t^{(n)}(x)) \ln|f_t(x) - \tilde{f}_t^{(n)}(x)| \beta(x) m(dx)$ tend to 0 as $n \to \infty$, cf. [18]. We use

$$Err_{3/2}(X(t), \tilde{X}^{(n)}(t)) := \left(\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^{3/2} m(dx) \right)^{2/3}$$

to measure the approximation error in this case.

Using the fact that $|x \ln x| \leq \max\{\sqrt{x}, x\sqrt{x}\}$ for $x > 0$ and that $-1 \leq \beta(x) \leq 1$ for all $x \in [-A, A]^d$, we have

$$
\left| \int_{[-A,A]^d} (f_t(x) - \tilde{f}_t^{(n)}(x))\beta(x) \ln |f_t(x) - \tilde{f}_t^{(n)}(x)| m(dx) \right|
$$

$$
\leq \int_{[-A,A]^d} \left| (f_t(x) - \tilde{f}_t^{(n)}(x))\beta(x) \ln |f_t(x) - \tilde{f}_t^{(n)}(x)| \right| m(dx)
$$

$$
\leq \int_{[-A,A]^d} \max\left\{ |f_t(x) - \tilde{f}_t^{(n)}(x)|^{1/2}, |f_t(x) - \tilde{f}_t^{(n)}(x)|^{3/2} \right\} m(dx)
$$

$$
\leq \int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^{1/2} m(dx) + \int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^{3/2} m(dx).
$$

Now, if $Err_{3/2}(X(t), \tilde{X}^{(n)}(t))$ tends to zero as n goes to infinity, then by Lyapunov's inequality (see [20]),

$$
\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^{3/2} m(dx) \to 0, \quad n \to \infty,
$$

$$
\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^{1/2} m(dx) \to 0, \quad n \to \infty,
$$

so that the integral $\int_{[-A,A]^d} (f_t(x) - \tilde{f}_t^{(n)}(x)) \ln |f_t(x) - \tilde{f}_t^{(n)}(x)|\beta(x) m(dx)$ goes to zero as $n \to \infty$. Furthermore, once again by using Lyapunov's inequality, the integral $\int_{[-A,A]^d} |f_t(x) - \tilde{f}_t^{(n)}(x)| m(dx)$ tends to zero as $n \to \infty$ so that $\tilde{X}^{(n)}(t)$ converges to $X(t)$ in probability.

Remark 3.5. If $Err_\alpha(X(t), \tilde{X}^{(n)}(t))$ tends to zero as n goes to infinity for each $t \in [-T, T]^q$, then

$$
\{\tilde{X}^{(n)}(t)\}_{t\in[-T,T]^q} \overset{f.d.}{\to} \{X(t)\}_{t\in[-T,T]^q},
$$

where $\overset{f.d.}{\to}$ denotes weak convergence of all finite dimensional distributions.

Proof. We first state a lemma which is an implication of the inequality

$$
(x + y)^p \leq x^p + y^p, \quad 0 < p \leq 1, \quad x, y > 0.
$$

Lemma 3.6. *Let $0 < p \leq 1$ and $f_i \in L^p(\mathbb{R}^d)$ for each $i \in \{1, \dots, n\}$ with $n \in \mathbb{N}$. Then*

$$
\left\| \sum_{i=1}^n f_i \right\|_{L^p}^p \leq \sum_{i=1}^n \|f_i\|_{L^p}^p.
$$

Now fix any $t_1, \dots, t_m \in [-T, T]^q$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. If $0 < \alpha < 1$, we get

$$
\left(Err_\alpha \left(\sum_{j=1}^m \lambda_j X(t_j), \sum_{j=1}^m \lambda_j \tilde{X}^{(n)}(t_j) \right) \right)^\alpha = \left\| \sum_{j=1}^m \lambda_j f_{t_j} - \sum_{j=1}^m \lambda_j \tilde{f}_{t_j}^{(n)} \right\|_{L^\alpha}^\alpha
$$

$$
\leq \sum_{j=1}^m |\lambda_j|^\alpha \left\| f_{t_j} - \tilde{f}_{t_j}^{(n)} \right\|_{L^\alpha}^\alpha = \sum_{j=1}^m |\lambda_j|^\alpha Err_\alpha(X(t_j), \tilde{X}^{(n)}(t_j))^\alpha \to 0, \quad n \to \infty.
$$

For $1 < \alpha \leq 2$, we can use Minkowski's inequality and get

$$Err_\alpha \left(\sum_{j=1}^{m} \lambda_j X(t_j), \sum_{j=1}^{m} \lambda_j \tilde{X}^{(n)}(t_j) \right) = \left\| \sum_{j=1}^{m} \lambda_j f_{t_j} - \sum_{j=1}^{m} \lambda_j \tilde{f}_{t_j}^{(n)} \right\|_{L^\alpha}$$

$$\leq \sum_{j=1}^{m} |\lambda_j| \left\| f_{t_j} - \tilde{f}_{t_j}^{(n)} \right\|_{L^\alpha} = \sum_{j=1}^{m} |\lambda_j| Err_\alpha(X(t_j), \tilde{X}^{(n)}(t_j)) \to 0, \quad n \to \infty.$$

Analogously for $\alpha = 1$,

$$Err_{3/2} \left(\sum_{j=1}^{m} \lambda_j X(t_j), \sum_{j=1}^{m} \lambda_j \tilde{X}^{(n)}(t_j) \right) \to 0, \quad n \to \infty.$$

Therefore, for any $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ we have $\sum_{j=1}^{m} \lambda_j X(t_j) - \sum_{j=1}^{m} \lambda_j \tilde{X}^{(n)}(t_j) \to 0$ in probability which implies the convergence of all fi-nite-dimensional distributions. $\square$

3.1.2. *Poisson random measures.* Let Φ be a Poisson random measure with intensity measure Θ and Ψ be the Poisson point process corresponding to the Poisson random measure. Furthermore, we assume that the kernel functions f_t are measurable on $[-A, A]^d$ for each $t \in [-T, T]^q$. Then, by the Campbell theorem (cf. [22]), we have

$$Err_1(X(t), \tilde{X}^{(n)}(t)) = \int_{[-A,A]^d} \left| f_t(x) - \tilde{f}_t^{(n)}(x) \right| \Theta(dx)$$

$$= \mathbb{E} \left(\int_{[-A,A]^d} \left| f_t(x) - \tilde{f}_t^{(n)}(x) \right| \Phi(dx) \right) = \mathbb{E} \left(\sum_{x \in \Psi} \left| f_t(x) - \tilde{f}_t^{(n)}(x) \right| \right)$$

$$\geq \mathbb{E} \left| \sum_{x \in \Psi} \left(f_t(x) - \tilde{f}_t^{(n)}(x) \right) \right| = \mathbb{E} \left| \int_{[-A,A]^d} f_t(x)\Phi(dx) - \int_{[-A,A]^d} \tilde{f}_t^{(n)}(x)\Phi(dx) \right|$$

$$= \mathbb{E} \left| X(t) - \tilde{X}^{(n)}(t) \right|,$$

that is we can control the mean error between $X(t)$ and $\tilde{X}^{(n)}(t)$ in the L^1-sense by $Err_1(X(t), \tilde{X}^{(n)}(t))$.

3.1.3. *Exploiting the spot variable representation of the second moment of $X(t)$.* We assume that the second moment of the random field exists and recall formula (2.8)

$$\mathbb{E}\left(X(t)^2 \right) = \int_{[-A,A]^d} f_t^2(y)\mathrm{Var}(L'(y))\lambda(dy) + \left(\int_{[-A,A]^d} f_t(y)\mathbb{E}(L'(y))\lambda(dy) \right)^2$$

which implies

$$
\begin{aligned}
\mathbb{E}(X(t) - \tilde{X}^{(n)}(t))^2 \;=\;& \int_{[-A,A]^d} (f_t(y) - \tilde{f}_t^{(n)}(y))^2 \mathrm{Var}(L'(y)) \lambda(dy) \\
&+ \left(\int_{[-A,A]^d} (f_t(y) - \tilde{f}_t^{(n)}(y)) \mathbb{E}(L'(y)) \lambda(dy) \right)^2 \\
\;\leq\;& \int_{[-A,A]^d} (f_t(y) - \tilde{f}_t^{(n)}(y))^2 \mathrm{Var}(L'(y)) \lambda(dy) \\
&+ \int_{[-A,A]^d} (f_t(y) - \tilde{f}_t^{(n)}(y))^2 \lambda(dy) \int_{[-A,A]^d} (\mathbb{E}(L'(y)))^2 \lambda(dy)
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality in the last inequality. If it holds $\mathrm{Var}(L'(y)) \leq c_1 < \infty$ for $y \in \mathbb{R}^d$ and $\int_{[-A,A]^d} (\mathbb{E}(L'(y)))^2 \lambda(dy) := c_2 < \infty$, then we get

$$
\left(\mathbb{E}(X(t) - \tilde{X}^{(n)}(t))^2 \right)^{1/2} \leq (c_1 + c_2)^{1/2} \| f_t - g_t \|_{L^2} = (c_1 + c_2)^{1/2} Err_2(X(t), \tilde{X}^{(n)}(t)).
$$

Example 3.7. Consider again the characteristic triplet $(a, 0, U)$ from Example 2.5. We have $\mathrm{Var}(L'(y)) = 1/\theta^2 =: c_1$ and $\int_{[-A,A]^d} (\mathbb{E}(L'(y)))^2 \lambda(dy) = (2A)^d/\theta^2 =: c_2$, so that

$$
\left(\mathbb{E}(X(t) - \tilde{X}^{(n)}(t))^2 \right)^{1/2} \leq \frac{1}{\theta} \left(1 + (2A)^d \right)^{1/2} Err_2(X(t), \tilde{X}^{(n)}(t)).
$$

3.2. **Step function approximation.** For any $n \in \mathbb{N}$ and $k = (k_1, \ldots, k_d)^\mathsf{T} \in \mathbb{Z}^d$ so that $-n \leq k_1, \ldots, k_d < n$, let

$$
\xi_k := \left(k_1 \frac{A}{n}, \ldots, k_d \frac{A}{n} \right)^\mathsf{T}, \quad \Delta_k := \left[k_1 \frac{A}{n}, (k_1 + 1) \frac{A}{n} \right) \times \cdots \times \left[k_d \frac{A}{n}, (k_d + 1) \frac{A}{n} \right).
$$

We define the step function $\tilde{f}_t^{(n)}(x) := \sum_{|k| \leq n} f_t(\xi_k) \mathbb{1}_{\Delta_k}(x)$ to approximate f_t, where $|k| \leq n$ means $-n \leq k_i < n$ for $i = 1, \ldots, d$. Then we have

$$
(3.4) \qquad \tilde{X}^{(n)}(t) = \int_{[-A,A]^d} \tilde{f}_t^{(n)}(x) \Lambda(dx) = \sum_{|k| \leq n} f_t(\xi_k) \Lambda(\Delta_k).
$$

The following theorem provides error bounds for $Err_s(X(t), \tilde{X}^{(n)}(t))$ for Hölder-continuous functions f_t.

Theorem 3.8. *Assume that $0 < s \leq 2$, the control measure λ is the Lebesgue measure and $f_t \in L^s(\mathbb{R}^d)$ is Hölder-continuous for all $t \in [-T,T]^q$, i. e.*

$$
|f_t(x) - f_t(y)| \leq C_t \cdot \|x - y\|_2^{\gamma_t}, \quad x, y \in [-A, A]^d,
$$

for and $t \in [-T,T]^q$, some $0 < \gamma_t \leq 1$ and $C_t > 0$, where $\|\cdot\|_2$ denotes the Euclidean norm. Then for any $t \in [-T,T]^q$ we have for all $n \geq 1$ that

$$
(3.5) \qquad Err_s(X(t), \tilde{X}^{(n)}(t)) \leq \left(\frac{2^d C_t d}{1 + \gamma_t s} \right)^{1/s} A^{\gamma_t + d/s} \left(\frac{1}{n} \right)^{\gamma_t}.
$$

Proof. Since

$$X(t) - \tilde{X}^{(n)}(t) = \int_{[-A,A]^d} \left(f_t(x) - \sum_{|k| \le n} f_t(\xi_k) \mathbb{1}_{\Delta_k}(x) \right) \Lambda(dx),$$

we have

$$\left(Err_s(X(t), \tilde{X}^{(n)}(t)) \right)^s = \int_{[-A,A]^d} \left| \sum_{|k| \le n} (f_t(x) - f_t(\xi_k)) \mathbb{1}_{\Delta_k}(x) \right|^s dx.$$

For each $x \in [-A, A]^d$, there exists exactly one index $\tilde{k} = \tilde{k}(x)$ with $|\tilde{k}| \le n$ and $x \in \Delta_{\tilde{k}}$. Hence

$$\left| \sum_{|k| \le n} (f_t(x) - f_t(\xi_k)) \mathbb{1}_{\Delta_k}(x) \right|^s = |f_t(x) - f_t(\xi_k)|^s \mathbb{1}_{\Delta_{\tilde{k}}}(x)$$

$$= \sum_{|k| \le n} |f_t(x) - f_t(\xi_k)|^s \mathbb{1}_{\Delta_k}(x)$$

which implies

$$\left(Err_s(X(t), \tilde{X}^{(n)}(t)) \right)^s = \int_{[-A,A]^d} \left| \sum_{|k| \le n} (f_t(x) - f_t(\xi_k)) \mathbb{1}_{\Delta_k}(x) \right|^s dx$$

$$= \sum_{|k| \le n} \int_{\Delta_k} |f_t(x) - f_t(\xi_k)|^s \, dx \le C_t \sum_{|k| \le n} \int_{\Delta_k} ||x - \xi_k||_2^{\gamma_t s} dx$$

$$= C_t \sum_{|k| \le n} \int_0^{A/n} \cdots \int_0^{A/n} \left(y_1^2 + \cdots + y_d^2 \right)^{(\gamma_t s)/2} dy_d \cdots dy_1.$$

As $(\gamma_t s)/2 \le 1$, we have $\left(y_1^2 + \cdots + y_d^2 \right)^{(\gamma_t s)/2} \le y_1^{\gamma_t s} + \cdots + y_d^{\gamma_t s}$ and hence

$$\int_0^{A/n} \cdots \int_0^{A/n} \left(y_1^2 + \cdots + y_d^2 \right)^{(\gamma_t s)/2} dy_d \cdots dy_1$$

(3.6)
$$\le d \left(\frac{A}{n} \right)^{d-1} \int_0^{A/n} y_1^{\gamma_t s} dy_1 = \frac{d}{\gamma_t s + 1} \left(\frac{A}{n} \right)^{d + \gamma_t s}.$$

Therefore, we get

$$Err_s(X(t), \tilde{X}^{(n)}(t)) \le \left(\frac{2^d C_t d}{1 + \gamma_t s} \right)^{1/s} A^{\gamma_t + d/s} \left(\frac{1}{n} \right)^{\gamma_t}.$$

$\square$

Remark 3.9. In many applications, it suffices to consider a control measure λ proportional to the Lebesgue measure (cf. Example 2.5), that is $\lambda(d\eta) = c \cdot d\eta$ for some $c > 0$. In this case, one has to multiply the upper bound in (3.5) by $c^{1/s}$. If the control measure λ is not the Lebesgue measure, then, in general, the integral

$$\int_0^{A/n} \cdots \int_0^{A/n} \left(y_1^2 + \cdots + y_d^2 \right)^{(\gamma_t s)/2} \lambda(d(y_1, \ldots, y_d))$$

in (3.6) cannot be calculated explicitly so that one would have to include it in the upper bound of the approximation error.

Remark 3.10. In the proof, one can estimate

$$\int_0^{A/n} \cdots \int_0^{A/n} \left(y_1^2 + \cdots + y_d^2\right)^{(\gamma_t s)/2} dy_d \cdots dy_1$$

alternatively by

$$\int_0^{A/n} \cdots \int_0^{A/n} \|y\|_2^{\gamma_t s} dy_d \cdots dy_1 \leq \frac{1}{2^d} \int_{\|y\|_2 \leq \frac{A}{n}\sqrt{d}} \|y\|_2^{\gamma_t s} dy$$

and calculate the last integral using polar coordinates. This yields

$$Err_s(X(t), \tilde{X}^{(n)}(t)) \leq \left(\frac{C_t \pi d^{(\gamma_t s + d)/2}}{\gamma_t s + d}\right)^{1/s} A^{\gamma_t + d/s} \left(\frac{1}{n}\right)^{\gamma_t} D(d, s),$$

where

$$D(d, s) := \begin{cases} 1, & d = 2, \\ 2, & d = 3, \\ \pi, & d = 4, \\ \pi^{d-3} \frac{\Gamma(3/2)}{\Gamma(d/2)}, & d \geq 5 \text{ odd}, \\ \pi^{d-7/2} \frac{\Gamma(3/2)}{\Gamma((d-1)/2)}, & d \geq 5 \text{ even}. \end{cases}$$

It is straightforward to show that this estimate is better than the one in Theorem 3.8 if

$$\frac{\pi}{2^d} d^{(\gamma_t s + d)/2 - 1} \cdot D(d, s) \cdot \frac{\gamma_t s + 1}{\gamma_t s + d} < 1.$$

Remark 3.11. Suppose that the conditions of Theorem 3.8 hold true. If the support of the function f_t is not compact, we first need to estimate $X(t) = \int_{\mathbb{R}^d} f_t(x)\Lambda(dx)$ by $X_K(t) = \int_{[-K,K]^d} f_t(x)\Lambda(dx)$. For $K > 0$ large enough, the approximation error is small since

$$Err_s(X(t), X_K(t)) = \left(\int_{\mathbb{R}^d \setminus [-K,K]^d} |f_t(x)|^s dx\right)^{1/s}$$

tends to zero as $K \to \infty$ and $f_t, \tilde{f}_t^{(n)} \in L^s([-A, A]^d, \lambda)$. Let $\varepsilon > 0$. If $1 \leq s \leq 2$, choose $K > 0$ so that $Err_s(X(t), X_K(t)) \leq \varepsilon/2$. We can apply Theorem 3.8 to $X_K(\cdot)$ which yields $Err_s(X_K(t), \tilde{X}_K^{(n)}(t)) \leq \varepsilon/2$ for $n \in \mathbb{N}$ large enough. Then

$$Err_s(X(t), \tilde{X}_K^{(n)}(t)) \leq Err_s(X(t), X_K(t)) + Err_s(X_K(t), \tilde{X}_K^{(n)}(t)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If $0 < s < 1$, then we choose $K > 0$ so large that $Err_s(X(t), X_K(t)) \leq \varepsilon^s/2$. Again, we can apply Theorem 3.8 to $X_K(\cdot)$ and get $Err_s(X_K(t), \tilde{X}_K^{(n)}(t)) \leq \frac{\varepsilon^s}{2}$ for $n \in \mathbb{N}$ large enough. Then

$$\begin{aligned} Err_s(X(t), \tilde{X}_K^{(n)}(t)) &\leq \left(Err_s(X(t), X_K(t)) + Err_s(X_K(t), \tilde{X}_K^{(n)}(t))\right)^{1/s} \\ &\leq \left(\frac{\varepsilon^s}{2} + \frac{\varepsilon^s}{2}\right)^{1/s} = \varepsilon. \end{aligned}$$

Remark 3.12. Theorem 3.8 provides a pointwise estimate of the approximation error for each $t \in [-T, T]^q$. We can obtain a uniform error bound as follows.

Assume that $\gamma := \inf_{t \in [-T,T]^q} \gamma_t > 0$. Then for each $t \in [-T, T]^q$, f_t is Hölder-continuous with parameters γ and some constant $C_t^* > 0$. Set $C := \sup_{t \in [,T]^d} C_t^*$ and assume $C < \infty$. Then $Err_s(X(t), \tilde{X}^{(n)}(t))$ can be estimated by (3.5) with C_t and γ_t replaced by C and γ.

One can also consider the integrated error

$$Err_s(X, \tilde{X}^{(n)}) := \int_{[-T,T]^q} Err_s(X(t), \tilde{X}^{(n)}(t)) dt$$

and multiply the error bound by $(2T)^q$.

Remark 3.13. Assume that $0 < s \leq 2$ and the functions f_t are differentiable with $\|\nabla f_t(x)\|_2 \leq C_t < \infty$ for all $x \in [-A, A]^d$ and $t \in [-T, T]^q$, where ∇f_t denotes the gradient of f_t. Then for any $t \in [-T, T]^q$, inequality (3.5) holds for all $n \geq 1$ with $\gamma_t = 1$, that is

$$Err_s(X(t), \tilde{X}^{(n)}(t)) \leq \left(\frac{2^d C_t d}{1 + s} \right)^{1/s} A^{1+d/s} \cdot \frac{1}{n}$$

since f_t is Hölder-continuous with constant C_t and exponent $\gamma_t = 1$.

3.3. **Approximation by wavelet series.**

3.3.1. *Series representation of kernel functions.* Let $s > 0$, $f_t \in L^s(\mathbb{R}^d)$, $t \in \mathbb{R}^q$, and let $\{\xi_i\}_{i \in I}$ be a basis for $L^s(\mathbb{R}^d)$, where I is an index set. Then f_t can be represented as

$$(3.7) \qquad\qquad f_t = \sum_{i \in I} a_i \cdot \xi_i$$

for certain constants $a_i \in \mathbb{R}$. In order to approximate f_t, one can truncate (3.7) so that it consists only of a finite number of summands. In [4], the trigonometric system is used to approximate the kernel function of certain stable random fields. In this paper, we will go another way and analyze whether a wavelet system may be appropriate for the simulation of random fields with an infinitely divisible random measure as integrator.

3.3.2. *Haar wavelets.* We will use the so-called *Haar basis* to approximate the kernel functions. For a detailed introduction into wavelets, see for example [7, 8, 24].

Definition 3.14. Consider $\varphi^{\mathrm{Haar}} : \mathbb{R}^d \to \mathbb{R}$ with $\varphi^{\mathrm{Haar}}(x) := 1/(2A)^{1/2} \cdot \mathbb{1}_{[-A,A]}(x)$ for $x \in \mathbb{R}^d$ and the corresponding *mother wavelet* $\Psi^{\mathrm{Haar}} : \mathbb{R}^d \to \mathbb{R}$ defined by $\Psi^{\mathrm{Haar}}(x) := \varphi^{\mathrm{Haar}}(2x + A) - \varphi^{\mathrm{Haar}}(2x - A)$. The resulting basis

$$\mathcal{H} := \{\varphi^{\mathrm{Haar}}\} \cup \{\Psi^{\mathrm{Haar}}_{j-2^k, k}\}_{k \in \mathbb{N}_0, 2^k \leq j \leq 2^{k+1}-1}$$

is called Haar basis for $L^2([-A, A])$, where

$$\Psi^{\mathrm{Haar}}_{j,k}(x) := 2^{k/2} \Psi^{\mathrm{Haar}}(2^k(x + A) - (1 + 2 \cdot j)A).$$

Let $\Psi^0 := \varphi^{\mathrm{Haar}}$, $\Psi^1 := \Psi^{\mathrm{Haar}}$ and E be the set of non-zero vertexes of the unit cube $[0, 1]^d$. Consider the functions $\Psi^e : \mathbb{R}^d \to \mathbb{R}$, $e = (e_1, \ldots, e_d)^\mathsf{T} \in E$, defined by $\Psi^e(x) := \Psi^e(x_1, \ldots, x_d) := \Psi^{e_1}(x_1) \cdots \Psi^{e_d}(x_d)$ for $x = (x_1, \ldots, x_d)^\mathsf{T} \in \mathbb{R}^d$. Let $a = (A, \ldots, A)^\mathsf{T}$, $c = (1, \ldots, 1)^\mathsf{T} \in \mathbb{R}^d$. Translation by $j = (j_1, \ldots, j_d)^\mathsf{T}$ and dilation

by 2^k yields $\Psi^e_{j-2^kc,k}(x) := 2^{kd/2}\Psi^e(2^k(x-a)-A(c+2j))$ with $2^k \leq j_i \leq 2^{k+1}-1$, $k \in \mathbb{N}_0$, $i = 1, \ldots, d$, $e \in E$, that form an orthonormal basis of $L^2([-A,A]^d)$. Then, each $f \in L^2([-A,A]^d)$ has the expansion

$$(3.8) \qquad f_t = (f_t, \Psi^*)\Psi^* + \sum_{e \in E}\sum_{k=0}^{\infty}\sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}}(f_t, \Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k},$$

in the sense of $L^2([-A,A]^d)$ convergence, where

$$(3.9) \qquad\qquad \Psi^*(x) \quad := \quad \frac{1}{(2A)^{d/2}}, \quad x \in [-A,A]^d,$$

$$(3.10) \qquad\qquad (f_t, \Psi^e_{j-2^kc,k}) \quad := \quad \int_{[-A,A]^d} f(x)\Psi^e_{j-2^kc,k}(x)dx.$$

It can be shown that any function $f \in L^p([-A,A]^d)$ for some $p > 1$ can be represented by a wavelet series of the form (3.8) in the sense of $L^p([-A,A]^d)$-convergence. However, there exist examples of functions for which (3.8) does not hold in particular for $p = 1$ (see [8]). Recall that we use

$$Err_s(X(t), \tilde{X}^{(n)}(t)) = \left(\int_{[-A,A]^d}|f_t(x) - \tilde{f}_t^{(n)}(x)|^s dx\right)^{1/s}$$

to measure the approximation quality of $\tilde{X}^{(n)}$ for some $s > 0$. Therefore, we restrict our setting to kernel functions $f_t \in L^p([-A,A]^d) \cap L^s([-A,A]^d)$ with $p > 1$.

As noticed, we want to use the expansion (3.8) in order to approximate the kernel functions f_t by truncating the (potentially) infinite sum to a finite number of summands. The goal is then to find an upper bound for the approximation error.

3.3.3. *Approximation by cutting off at a certain detail level.* Let f_t belong to the space $L^p([-A,A]^d) \cap L^s([-A,A]^d)$, $p > 1$, be a kernel function with corresponding Haar series

$$f_t = (f_t, \Psi^*)\Psi^* + \sum_{e \in E}\sum_{k=0}^{\infty}\sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}}(f_t, \Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}.$$

The idea is now to cut off this series at a certain detail level $k = n$, that is to approximate the kernel function f_t by

$$\tilde{f}_{t,cut}^{(n)} = (f_t, \Psi^*)\Psi^* + \sum_{e \in E}\sum_{k=0}^{n}\sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}}(f_t, \Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}.$$

The following lemma provides an upper bound for the approximation error of bounded kernel functions by applying the cut-off truncation method.

Lemma 3.15. *Let $s > 0$ and $d > s$. Assume that $M_t := \sup_{x \in [-A,A]^d}|f_t(x)| < \infty$. Then for $n \in \mathbb{N}_0$*

$$\|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} \leq \begin{cases} \left(\frac{(2^d-1)}{2^{d-1}-1}\right)^{\frac{1}{s}}d^{\frac{1}{s}}M_t(2A)^{\frac{d}{s}}\left(\frac{1}{2^{(d-1)/s}}\right)^n, & 0 < s < 1, \\ \frac{2^d-1}{2^{d/s-1}-1} \cdot d \cdot M_t \cdot (2A)^{\frac{d}{s}} \cdot \left(\frac{1}{2^{d/s-1}}\right)^n, & s \geq 1. \end{cases}$$

Proof. Let $s \geq 1$. We have

$$(3.11) \qquad \|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} \leq \sum_{e \in E} \sum_{k=n+1}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}} |(f_t, \Psi_{j-2^k c,k}^e)| \left\| \Psi_{j-2^k c,k}^e \right\|_{L^s}.$$

Now $|(f_t, \Psi_{j-2^k c,k}^e)|$ can be estimated by

$$|(f_t, \Psi_{j-2^k c,k}^e)| \leq M_t \int_{[-A,A]^d} |\Psi_{j-2^k c,k}^e(x)| dx = M_t 2^{-kd/2} (2A)^{d/2}$$

and $\left\| \Psi_{j-2^k c,k}^e \right\|_{L^s}$ is equal to

$$\left\| \Psi_{j-2^k c,k}^e \right\|_{L^s} = \left(\int_{[-A,A]^d} |\Psi_{j-2^k c,k}^e(x)|^s dx \right)^{1/s} = \frac{2^{kd/2}}{(2A)^{d/2}} \cdot 2^{-kd/s} \cdot (2A)^{d/s}.$$

Thus

$$\|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} \leq \sum_{e \in E} \sum_{k=n+1}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}} M_t 2^{-\frac{kd}{2}} 2^{\frac{kd}{2}} 2^{-\frac{kd}{s}} (2A)^{\frac{d}{s}}$$

$$= (2^d - 1) M_t d (2A)^{d/s} \sum_{k=n+1}^{\infty} \left(2^{1-d/s} \right)^k.$$

Since $d > s$, we have $1 - d/s < 0$ and the geometric series formula implies

$$(3.12) \qquad \|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} \leq \frac{2^d - 1}{2^{d/s-1} - 1} d M_t (2A)^{d/s} \left(\frac{1}{2^{d/s-1}} \right)^n.$$

Now let $0 < s < 1$. By Lemma 3.6, we have

$$\left\| f_t - \tilde{f}_{t,cut}^{(n)} \right\|_{L^s}^s \leq \sum_{e \in E} \sum_{k=n+1}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}} |(f_t, \Psi_{j-2^k c,k}^e)|^s \left\| \Psi_{j-2^k c,k}^e \right\|_{L^s}^s.$$

By using the estimates for the wavelet coefficients and the L^s-norms of the wavelets from above, we get

$$\left\| f_t - \tilde{f}_{t,cut}^{(n)} \right\|_{L^s}^s \leq (2^d - 1) d \left(M_t (2A)^{d/s} \right)^s \sum_{k=n+1}^{\infty} \left(2^{1-d} \right)^k$$

and finally

$$\|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} \leq \frac{(2^d - 1)^{1/s}}{(2^{d-1} - 1)^{1/s}} \cdot d^{1/s} \cdot M_t \cdot (2A)^{d/s} \cdot \left(\frac{1}{2^{(d-1)/s}} \right)^n.$$

$\square$

If we make further assumptions about the kernel function f_t, we can improve the rate of convergence of the upper bound.

Lemma 3.16. *Assume that f_t is Hölder-continuous with parameters C_t and γ_t for all $t \in [-T,T]^q$. Then for $n \in \mathbb{N}_0$*

$$\|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} \leq \begin{cases} \left(\frac{2^d-1}{2^{d+s\gamma_t-1}-1}\right)^{\frac{1}{s}} \frac{d^{\frac{1}{s}}2 + \frac{\gamma_t}{2}}{C}_t(2A)^{\frac{d}{s}+\gamma_t}\left(\frac{1}{2^{\frac{d-1}{s}+\gamma_t}}\right)^n, & 0 < s < 1, \\ \frac{2^d-1}{2^{d/s+\gamma_t}-2} \cdot d^{1+\gamma_t/2}C_t(2A)^{d/s+\gamma_t}\left(\frac{1}{2^{d/s+\gamma_t-1}}\right)^n, & s \geq 1. \end{cases}$$

Proof. We estimate $|(f_t, \Psi_{j-2^k c,k}^e)|$ as in the preceding lemma. Let

$$B := \{x \in [-A,A]^d : \Psi_{j-2^k c,k}^e(x) > 0\} \text{ and } C := \{x \in [-A,A]^d : \Psi_{j-2^k c,k}^e(x) < 0\}.$$

Then we have

$$|(f_t, \Psi_{j-2^k c,k}^e)| = \left| \int_{[-A,A]^d} f_t(x)\Psi_{j-2^k c,k}^e(x)dx \right|$$

$$= \left| \int_B f_t(x)\frac{2^{kd/2}}{(2A)^{d/2}}dx - \int_C f_t(x)\frac{2^{kd/2}}{(2A)^{d/2}}dx \right|$$

$$\leq \frac{2^{\frac{kd}{2}}}{(2A)^{\frac{d}{2}}} \max\left\{ \left| \max_{x\in B} f_t(x)|B| - \max_{x\in C} f_t(x)|C| \right|, \left| \min_{x\in B} f_t(x)|B| - \max_{x\in C} f_t(x)|C| \right|, \right.$$

$$\left. \left| \max_{x\in B} f_t(x)|B| - \min_{x\in C} f_t(x)|C| \right|, \left| \min_{x\in B} f_t(x)|B| - \min_{x\in C} f_t(x)|C| \right| \right\}.$$

Here, $|B|$ and $|C|$ denotes the volume of B and C, respectively. Let $x_1 \in B$ and $x_2 \in C$ so that the maximum in the last inequality is attained. Furthermore, we have $|B| = |C| = 1/2 \cdot (2A)^d \cdot 2^{-kd}$. Then, since f_t is Hölder-continuous, we get

$$|(f_t, \Psi_{j-2^k c,k}^e)| \leq \frac{2^{kd/2}}{(2A)^{d/2}} |f_t(x_1) \cdot |B| - f_t(x_2) \cdot |C||$$

$$= \frac{2^{kd/2}}{(2A)^{d/2}} \cdot \frac{1}{2}(2A)^d \cdot 2^{-kd}|f_t(x_1) - f_t(x_2)| \leq \frac{1}{2}(2A)^{d/2} \cdot 2^{-kd/2}C_t\|x_1 - x_2\|^{\gamma_t}$$

$$\leq \frac{1}{2}(2A)^{d/2} \cdot 2^{-kd/2}C_t\left(2A\sqrt{d}2^{-k}\right)^{\gamma_t} = A^{d/2+\gamma_t}C_t 2^{d/2+\gamma_t-1}d^{\gamma_t/2}\left(\frac{1}{2^{d/2+\gamma_t}}\right)^k.$$

The remainder of the proof is analogous to the one of Lemma 3.15. $\qquad\square$

Corollary 3.17. *Assume that f_t is differentiable with $\|\nabla f_t(x)\|_2 \leq C_t$ for all $x \in [-A,A]^d$, some $C_t > 0$ and $t \in [-T,T]^q$. Then for $n \in \mathbb{N}_0$*

$$\|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} \leq \begin{cases} \frac{1}{2}\left(\frac{2^d-1}{2^{d+s-1}-1}\right)^{\frac{1}{s}} d^{\frac{1}{s}+\frac{1}{2}}C_t(2A)^{\frac{d}{s}+1}\left(\frac{1}{2^{(d-1)/s+1}}\right)^n, & 0 < s < 1, \\ \frac{2^d-1}{2^{d/s+1}-2} \cdot d^{3/2} \cdot C_t \cdot (2A)^{d/s+1} \cdot \left(\frac{1}{2^{d/s}}\right)^n, & s > 1. \end{cases}$$

3.3.4. *Near best n-term approximation.* For $p > 1$ and $s > 0$, taking a wavelet basis of $L^p([-A,A]^d) \cap L^s([-A,A]^d)$ has advantages in particular in the representation of functions with discontinuities and sharp peaks, that is functions with a certain local behavior. By simply cutting of at a certain detail level, this advantage is not honored. In view of (3.11), we may expect that we have to calculate less Haar coefficients if we approximate the kernel function f_t by a truncated Haar series $\tilde{f}_t^{(n)}$ that contains those n summands $(f, \Psi_{j-2^k c,k}^e)\Psi_{j-2^k c,k}^e$ with the largest values

$$\left\| (f_t, \Psi_{j-2^k c,k}^e)\Psi_{j-2^k c,k}^e \right\|_{L^s} = |(f_t, \Psi_{j-2^k c,k}^e)| \left\| \Psi_{j-2^k c,k}^e \right\|_{L^s}.$$

An approach to use such a truncation in order to approximate functions is presented in [7]. We summarize the main statements.

Consider a function S defined by $S = \sum_{(e,k,j)\in\Xi} a^e_{j,k}\Psi^e_{j-2^k c,k}$, where

$$\Xi \;=\; \{(e,k,j) : e\in E, k\in\mathbb{N}_0, 2^k \le j_i \le 2^{k+1}-1, i=1,\dots,d\},$$

the cardinality of Ξ is less or equal to $n\in\mathbb{N}$ and $a^e_{j,k}\in\mathbb{R}$ for all $(e,j,k)\in\Xi$. Hence, (3.3.4) is a linear combination of n Haar wavelets. We denote by Σ_n the set of all functions S defined as in (3.3.4) and let $\sigma^s_n(f_t) := \inf_{S\in\Sigma_n} \|f_t - S\|_{L^s}$. Now, we truncate the wavelet expansion (3.8) of f_t by taking those n summands for which the absolute value of $|(f_t, \Psi^e_{j-2^k c,k})| \left\|\Psi^e_{j-2^k c,k}\right\|_{L^s}$ is largest and denote the truncated sum by $\tilde{f}_t^{(n)}$. In [23], the following theorem has been proven which shows that this truncation is a *near best n-term approximation* (in the sense of (3.13)).

Theorem 3.18. *Let $1 < s < \infty$. Then for any function $f \in L^s([-A, A]^d)$, we have*

$$(3.13) \qquad\qquad \|f - \tilde{f}^{(n)}\|_{L^s} \le C_1(s, d, A)\sigma^s_n(f_t)$$

with a constant $C_1(s, d, A) \ge 0$ only depending on s, d and A.

If the sequence $\left\{|(f, \Psi^e_{j-2^k c,k})| \left\|\Psi^e_{j-2^k c,k}\right\|_{L^s}\right\}_{j,k}$ is in the Lorentz space wl_τ, $0 < \tau < \infty$, that is

$$(3.14) \qquad\qquad \#\{(j,k,e) : \{\|(f, \Psi^e_{j,k})\Psi^e_{j,k}\|_{L^s} > \varepsilon\} \le \left(\frac{M}{\varepsilon}\right)^\tau$$

for any $\varepsilon > 0$ and some $M \ge 0$ with a certain additional condition on τ, then $\sigma^s_n(f_t)$ can be bounded from above as shown in [7].

Theorem 3.19. *Let $1 < s < \infty$ and $f \in L^s([-A, A]^d)$. Furthermore, let*

$$\left\{\left\|(f, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}\right\|_{L^s}\right\}_{j,k} \in wl_\tau$$

and $u > 0$ with $1/\tau = u + 1/s$. Then

$$\sigma^s_n(f) \le C_2(s, d, A)M\left(\frac{1}{n}\right)^u, \quad n = 1, 2, \dots,$$

with a constant $C_2(s, d, A) \ge 0$ only depending on s, d and A and M being a constant satisfying (3.14).

Remark 3.20. As mentioned above, we cannot use the Haar basis for $L^s([-A, A]^d)$ if $s \le 1$ because there exist functions $f \in L^s([-A, A]^d)$ which cannot be represented by a wavelet series of the form (3.8). In this case, the Haar basis has to be replaced by a smoother wavelet basis and the so-called *Hardy space* $H^s([-A, A]^d)$ has to be used in place of $L^s([-A, A]^d)$. Then any function $f \in H^s([-A, A]^d)$ has a wavelet expansion for a certain range of $0 < s \le 1$ that depends on the smoothness of the underlying wavelet basis (see [8]).

Combining Theorem 3.18 and Theorem 3.19 yields an upper bound of the approximation error by using the near best n-term approximation with a rate of convergence of $O\left((1/n)^u\right)$. In [13], the following formulas for C_1 and C_2 are derived

for functions $f \in L^s([0,1]^d)$ (instead of $f \in L^s([-A,A]^d)$):

$$
\begin{aligned}
C_1(s,d) &= \left(2 + \left(1 - 2^{-\frac{d}{s}}\right)^{-2}\right)\left((2^d - 1)\left(\max\left(s, \frac{s}{s-1}\right) - 1\right)\right)^2, \\
C_2(s,d) &= \frac{2\left(2^{\tau/s} - 1\right)}{\left(1 - \left(\frac{1}{2}\right)^{d/s}\right)\left(1 - \left(\frac{1}{2}\right)^{\tau}\right)^{1/s}\left(1 - 2^{\tau/s-1}\right)}.
\end{aligned}
$$

We notice that on the one hand, these constants are not sharp and may be quite large, and on the other hand, we would have to find a value of M in (3.14) as small as possible for each kernel function f_t or for certain classes of kernel functions, which is not so easy to determine. Therefore, we suggest an approach in the following which is not based on the error estimate with those constants, but still determines an approximation at least close to the near best n-term approximation while keeping the desired level of accuracy.

If $s > 1$, it is clear that $\|(f, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}\|_{L^s}$ goes to zero as $k \to \infty$. This means that the n largest values $\|(f, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}\|_{L^s}$ are likely to be found for small values of k.

We now assume that $d > s$ and $M_t := \sup_{x \in [-A,A]^d} |f_t(x)| < \infty$ and choose $\varepsilon > 0$ as the desired level of accuracy.

Let $s > 1$. From Lemma 3.15 and its proof, we get

$$
\|f_t - \tilde{f}^{(n)}_{t,cut}\|_{L^s} \le \frac{2^d - 1}{2^{d/s-1} - 1} \cdot d \cdot M_t \cdot (2A)^{d/s} \cdot \left(\frac{1}{2^{d/s-1}}\right)^n \le \varepsilon
$$

if

$$
n \ge \frac{\ln(\varepsilon(2^{d/s-1} - 1)) - \ln((2^d - 1)dM_t(2A)^{d/s})}{(1 - d/s)\ln 2}.
$$

We take

$$
m_t := \left\lceil \frac{\ln(\varepsilon(2^{d/s-1} - 1)) - \ln((2^d - 1)dM_t(2A)^{d/s})}{(1 - d/s)\ln 2} \right\rceil,
$$

where $\lceil x \rceil$ is the integral part of x, as our minimal detail level to obtain the desired level of accuracy.

If $0 < s \le 1$, then by analogous calculations

$$
(3.15) \qquad m_t := \left\lceil \frac{\ln\left(\varepsilon(2^{d-1} - 1)^{\frac{1}{s}}\right) - \frac{1}{s}\ln\left((2^d - 1)dM_t^s(2A)^d\right)}{\frac{d-1}{s}\ln 2} \right\rceil.
$$

We now add $l \in \mathbb{N}_0$ detail levels to the truncated wavelet series at detail level m_t, that is we consider

$$
(3.16) \qquad \tilde{f}^{(m_t+l)}_{t,cut} = (f_t, \Psi^*)\Psi^* + \sum_{e \in E} \sum_{k=0}^{m_t+l} \sum_{\substack{2^k \le j_i \le 2^{k+1}-1 \\ i=1,\ldots,d}} (f_t, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}.
$$

Furthermore, we define

$$
(3.17) \quad C := \|(f_t, \Psi^*)\Psi^*\|_{L^s} + \sum_{e \in E} \sum_{k=0}^{m_t+l} \sum_{\substack{2^k \le j_i \le 2^{k+1}-1 \\ i=1,\ldots,d}} \|(f_t, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}\|_{L^s}
$$

and

$$D := \|(f_t, \Psi^*)\Psi^*\|_{L^s} + \sum_{e \in E} \sum_{k=m_t+l}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\dots,d}} \|(f_t, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}\|_{L^s}.$$

By Lemma 3.15, the corresponding level of accuracy of (3.16) is at least

$$\varepsilon_t^* := \begin{cases} \frac{2^d-1}{2^{d/s-1}-1} \cdot d \cdot M_t \cdot (2A)^{d/s} \cdot \left(\frac{1}{2^{d/s-1}}\right)^{m_t+l}, & s > 1, \\ \left(\frac{2^d-1}{2^{d-1}-1}\right)^{1/s} \cdot d^{1/s} \cdot M_t \cdot (2A)^{d/s} \cdot \left(\frac{1}{2^{(d-1)/s}}\right)^{m_t+l}, & 0 < s \leq 1. \end{cases}$$

Now we take the n largest summands from (3.17), that is from

$$\{\|(f_t, \Psi^*)\Psi^*\|_{L^s}\} \cup \{\|(f_t, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}\|_{L^s}\}_{j,k}$$

and denote them by $a_1, \dots, a_n$. The remaining summands are denoted by a_{n+1}, $a_{n+2}, \dots$, and the corresponding summands from equation (3.16) by $b_1, \dots, b_n$, $b_{n+1}, b_{n+2}, \dots$. The number $n \in \mathbb{N}$ is chosen to be the smallest number so that

$$C - \sum_{i=1}^{n} a_i \leq \varepsilon - \varepsilon_t^*.$$

We define $\tilde{f}_t^{(n)} := \sum_{i=1}^{n} b_i$ which is close to the near best n-term approximation if $s > 1$ and l is chosen large enough since $\|(f_t, \Psi^e_{j-2^{m_t+l},m_t+l})\Psi^e_{j-2^{m_t+l},m_t+l}\|_{L^s}$ goes to zero as l goes to infinity. Then we have

$$\begin{aligned} \|f_t - \tilde{f}_t^{(n)}\|_{L^s} &= \left\|\sum_{i=n+1}^{\infty} b_i\right\|_{L^s} \leq \sum_{i=n+1}^{\infty} a_i \\ &= C - \sum_{i=1}^{n} a_i + D \leq \varepsilon - \varepsilon_t^* + \varepsilon_t^* = \varepsilon. \end{aligned}$$

3.3.5. *Implementation.* When implementing the wavelet approach, one more problem has to be considered which we discuss now. Let I be the set of the indexes (e, j, k) for which the summands $(f_t, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}$ are part of the approximation $\tilde{f}_t^{(n)}$. Then we can write

$$\tilde{f}_t^{(n)} = (f_t, \Psi^*)\Psi^* + \sum_{(e,k,j) \in I} (f_t, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}$$

if $(f_t, \Psi^*)\Psi^*$ is included in the truncated series or

$$\tilde{f}_t^{(n)} = \sum_{(e,k,j) \in I} (f_t, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}$$

if it is not included. In order to approximate the random field X, we use

$$\tilde{X}^{(n)}(t) = (f_t, \Psi^*) \cdot \frac{\Lambda([-A,A]^d)}{(2A)^{d/2}} + \sum_{(e,k,j) \in I} (f_t, \Psi^e_{j-2^k c,k}) \int_{[-A,A]^d} \Psi^e_{j-2^k c,k} \Lambda(dx)$$

or

$$\tilde{X}^{(n)}(t) = \sum_{(e,k,j) \in I} (f_t, \Psi^e_{j-2^k c,k}) \int_{[-A,A]^d} \Psi^e_{j-2^k c,k} \Lambda(dx)$$

if again $(f_t, \Psi^*)\Psi^*$ is not included in the truncated series. Since the Haar wavelets $\Psi^e_{j-2^k c,k}$ are simple step functions, the integrals $\int_{[-A,A]^d} \Psi^e_{j-2^k c,k}\Lambda(dx)$ can be easily simulated although they are not independent: Let $z \in \mathbb{N}$ be the finest detail level of the wavelet approximation. Then all of these integrals can be built up from

$$\int_{[-A+k\frac{A}{2^z}, -A+(k+1)\frac{A}{2^z})^d} \Lambda(dx) = \Lambda\left(\left[-A+k\frac{A}{2^z}, -A+(k+1)\frac{A}{2^z}\right)^d\right)$$

for $k = 0, \ldots, 2^{z+1} - 1$. These random variables are independent because the sets $[-A+k\frac{A}{2^z}, -A+(k+1)\frac{A}{2^z})^d$ are disjoint.

However, the wavelet coefficients $(f_t, \Psi^e_{j-2^k c,k})$ cause problems if no closed formula of the integral of the kernel functions f_t over cubes is known. In this case, they have to be determined numerically by using the fast wavelet transform (see for instance [24]).

This results in a further approximation error which we need to estimate. When the detail level at which the wavelet series is cut off is equal to n, the input vector of the fast wavelet transform consists of integrals of the form

$$(3.18) \qquad \int_{cube} \frac{2^{(n+1)d/2}}{(2A)^{d/2}} f_t(x)dx,$$

where $cube$ is a cube in $\mathbb{R}^d$ of side length $2^{-d(n+1)}$.

We now assume that we have calculated these integrals with a precision of $\delta > 0$ and denote the wavelet coefficients computed by the fast wavelet transform by $\widehat{(f_t, \Psi^*)}$ and $(f_t, \widehat{\Psi^e_{j-2^k c,k}})$. When applying the fast wavelet transform, the value of each integral is used $2^d - 1$ times at each detail level k to calculate the wavelet coefficients $(f_t, \Psi^e_{j-2^k c,k})$ for any $e \in E$. For (f_t, Ψ^*), each integral is used once. There are $2^{(n+1)d}$ such integrals. Furthermore, the wavelet coefficients $(f_t, \Psi^e_{j-2^k c,k})$ consist of integrals of the form $\int_{cube} 2^{kd/2}/(2A)^{d/2} f_t(x)dx$. Since (3.18) contains the factor $2^{(n+1)d/2}/(2A)^{d/2}$, the factors $2^{kd/2}/2^{(n+1)d/2}$ are used in the following estimates for $|(f_t, \widehat{\Psi^e_{j-2^k c,k}}) - (f_t, \Psi^e_{j-2^k c,k})|$ to adjust for this fact.

When $s \geq 1$, the precision of

$$\widehat{f^{(n)}_{t,cut}} = \widehat{(f_t, \Psi^*)}\Psi^* + \sum_{e \in E} \sum_{k=0}^{n} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}} (f_t, \widehat{\Psi^e_{j-2^k c,k}})\Psi^e_{j-2^k c,k}$$

to approximate

$$(3.19) \quad \tilde{f}^{(n)}_{t,cut} = (f_t, \Psi^*)\Psi^* + \sum_{e \in E} \sum_{k=0}^{n} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}} (f_t, \Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}$$

is

$$\|\widehat{\tilde{f}_{t,cut}^{(n)}} - \tilde{f}_{t,cut}^{(n)}\|_{L^s}$$

$$\leq \quad |(\widehat{f_t, \Psi^*}) - (f_t, \Psi^*)| \|\Psi^*\|_{L^s}$$

$$+ \sum_{e \in E} \sum_{k=0}^{n} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\dots,d}} |(f_t, \widehat{\Psi_{j-2^k c,k}^e}) - (f_t, \Psi_{j-2^k c,k}^e)| \|\Psi_{j-2^k c,k}^e\|_{L^s}$$

$$\leq \quad 2^{(n+1)d} \frac{(2A)^{d/s-d/2}}{2^{(n+1)d/2}} \delta + (2^d - 1) \sum_{k=0}^{n} 2^{(n+1)d} \frac{2^{kd/2}}{2^{(n+1)d/2}} \delta (2A)^{\frac{d}{s}-\frac{d}{2}} 2^{\frac{kd}{2}-\frac{kd}{s}}$$

$$\leq \quad \begin{cases} (2A)^{\frac{d}{s}-\frac{d}{2}} 2^{\frac{(n+1)d}{2}} \left(1 + (2^d-1)\frac{2^{(d-\frac{d}{s})(n+1)}-1}{2^{d-d/s}-1}\right) \delta, & s > 1, \\ (2A)^{d/2} 2^{(n+1)d/2} \left(1 + (2^d-1)(n+1)\right) \delta, & s = 1. \end{cases}$$

When $0 < s < 1$, we can use Lemma 3.6 and get

$$\|\widehat{\tilde{f}_{t,cut}^{(n)}} - \tilde{f}_{t,cut}^{(n)}\|_{L^s}$$

$$\leq \quad \left(|(\widehat{f_t, \Psi^*}) - (f_t, \Psi^*)|^s \|\Psi^*\|_{L^s}^s \right.$$

$$\left. + \sum_{e \in E} \sum_{k=0}^{n} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\dots,d}} \left|(f_t, \widehat{\Psi_{j-2^k c,k}^e}) - (f_t, \Psi_{j-2^k c,k}^e)\right|^s \|\Psi_{j-2^k c,k}^e\|_{L^s}^s \right)^{\frac{1}{s}}$$

$$\leq \quad (2A)^{d/s-d/2} 2^{(n+1)d(1-s/2)} \left(1 + (2^d-1)\frac{2^{(ds-d)(n+1)}-1}{2^{ds-d}-1}\right)^{1/s} \delta.$$

Let $\varepsilon > 0$. We choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so that $\varepsilon_1 + \varepsilon_2 = \varepsilon$ if $s \geq 1$ and $\varepsilon_1^s + \varepsilon_2^s = \varepsilon^s$ if $0 < s < 1$. Furthermore, we choose the detail level n so large (by using the formulas in Section 3.3.3) such that $\|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} \leq \varepsilon_1$. We approximate the elements of the input vector for the fast wavelet transform with a precision of

$$\delta = \begin{cases} \dfrac{\varepsilon_2}{(2A)^{\frac{d}{s}-\frac{d}{2}} 2^{\frac{(n+1)d}{2}} \left(1+(2^d-1)\frac{2^{(ds-d)(n+1)}-1}{2^{ds-d}-1}\right)^{1/s}}, & 0 < s < 1, \\[3ex] \dfrac{\varepsilon_2}{(2A)^{d/2} 2^{(n+1)d/2} (1+(2^d-1)(n+1))}, & s = 1, \\[3ex] \dfrac{\varepsilon_2}{(2A)^{\frac{d}{s}-\frac{d}{2}} 2^{(n+1)d(1-\frac{s}{2})} \left(1+(2^d-1)\frac{2^{(d-\frac{d}{s})(n+1)}-1}{2^{d-\frac{d}{s}}-1}\right)}, & s \geq 1. \end{cases}$$

Then we have for $s \geq 1$

$$\|f_t - \widehat{\tilde{f}_{t,cut}^{(n)}}\|_{L^s} \leq \|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s} + \|\tilde{f}_{t,cut}^{(n)} - \widehat{\tilde{f}_{t,cut}^{(n)}}\|_{L^s} \leq \varepsilon_1 + \varepsilon_2 = \varepsilon,$$

and for $0 < s < 1$

$$\|f_t - \widehat{\tilde{f}_{t,cut}^{(n)}}\|_{L^s} \leq \left(\|f_t - \tilde{f}_{t,cut}^{(n)}\|_{L^s}^s + \|\tilde{f}_{t,cut}^{(n)} - \widehat{\tilde{f}_{t,cut}^{(n)}}\|_{L^s}^s \right)^{1/s} = \varepsilon.$$

We summarize this result in the following algorithm.

Algorithm

Let $M_t := \sup_{x \in [-A,A]^d} |f_t(x)| < \infty$ and $d > s$. Choose $\varepsilon > 0$ as the desired level of accuracy. Choose $\varepsilon_1, \varepsilon_2 > 0$ so that $\varepsilon = \varepsilon_1 + \varepsilon_2$ if $s \geq 1$ and $\varepsilon = (\varepsilon_1^s + \varepsilon_2^s)^{1/s}$ if $0 < s < 1$.

(a) Let

$$
m_t := \begin{cases} \left\lceil \dfrac{\ln\left(\varepsilon(2^{d-1}-1)^{1/s}\right) - \frac{1}{s}\ln\left((2^d-1)dM_t^s(2A)^d\right)}{\frac{d-1}{s}\ln 2} \right\rceil, & 0 < s < 1, \\[3mm] \left\lceil \dfrac{\ln(\varepsilon(2^{d/s-1}-1)) - \ln((2^d-1)dM_t(2A)^{d/s})}{(1-d/s)\ln 2} \right\rceil, & s \geq 1, \end{cases}
$$

and choose a number $l \in \mathbb{N}_0$ that increases the detail level m_t.

(b) Calculate the wavelet coefficients for

$$
(f_t, \Psi^*)\Psi^* + \sum_{e \in E} \sum_{k=0}^{m_t+l} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}} (f_t, \Psi_{j-2^k c,k}^e)\Psi_{j-2^k c,k}^e
$$

using the fast wavelet transform with a precision of

$$
\delta = \begin{cases} \dfrac{\varepsilon_2(2A)^{d/2-d/s}2^{-(n+1)d/2}}{\left(1+(2^d-1)\frac{2^{(ds-d)(n+1)}-1}{2^{ds-d}-1}\right)^{1/s}}, & 0 < s < 1, \\[3mm] \dfrac{\varepsilon_2(2A)^{-d/2}2^{-(n+1)d/2}}{(1+(2^d-1)(n+1))}, & s = 1, \\[3mm] \dfrac{\varepsilon_2(2A)^{d/s-d/2}2^{(n+1)d(1-s/2)}}{\left(1+(2^d-1)\frac{2^{(d-d/s)(n+1)}-1}{2^{d-d/s}-1}\right)}, & s > 1. \end{cases}
$$

(c) Take the n largest summands from

$$
C := \|(\widehat{f_t, \Psi^*})\Psi^*\|_{L^s} + \sum_{e \in E} \sum_{k=0}^{m_t+l} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}} \left\| (f_t, \widehat{\Psi_{j-2^k c,k}^e})\Psi_{j-2^k c,k}^e \right\|_{L^s}
$$

and denote them by $a_1, \ldots, a_n$. The corresponding summands from

$$
(\widehat{f_t, \Psi^*})\Psi^* + \sum_{e \in E} \sum_{k=0}^{m_t+l} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\ldots,d}} (f_t, \widehat{\Psi_{j-2^k c,k}^e})\Psi_{j-2^k c,k}^e
$$

are denoted by $b_1, \ldots, b_n$. Choose the number n so that $C - \sum_{i=1}^{n} a_i \leq \varepsilon_1 - \varepsilon_t^*$, where

$$
\varepsilon_t^* = \begin{cases} \left(\dfrac{2^d-1}{2^{d-1}-1}\right)^{\frac{1}{s}} d^{1/s} M_t(2A)^{d/s} \left(\dfrac{1}{2^{(d-1)/s}}\right)^{m_t+l}, & 0 < s < 1, \\[3mm] \dfrac{2^d-1}{2^{d/s-1}-1} dM_t(2A)^{d/s} \left(\dfrac{1}{2^{d/s-1}}\right)^{m_t+l}, & s \geq 1. \end{cases}
$$

(d) Take $\tilde{f}_t^{(n)} = \sum_{i=1}^{n} b_i$ as the approximation for f_t.

Remark 3.21.   (a) Assume that $d \in \mathbb{N}$ and the kernel functions f_t are Hölder-continuous with parameters C_t and γ_t for each $t \in [-T, T]^q$. Then the algorithm can be applied with m_t and ε_t^* replaced by

$$m_t := \begin{cases} \left\lceil \frac{\ln(2\varepsilon_1(2^{d+s\gamma_t-1}-1)^{1/s})}{((1-d)/s-\gamma_t)\ln 2} - \frac{\ln((2^d-1)^{1/s}d^{1/s+\gamma_t/(2s)}C_t(2A)^{d/s+\gamma_t})}{((1-d)/s-\gamma_t)\ln 2} \right\rceil, & 0 < s < 1, \\ \left\lceil \frac{\ln(\varepsilon_1(2^{d/s+\gamma_t}-2))}{(1-d/s-\gamma_t)\ln 2} - \frac{\ln((2^d-1)d^{1+\gamma_t/2}C_t(2A)^{d/s+\gamma_t})}{(1-d/s-\gamma_t)\ln 2} \right\rceil, & s \geq 1. \end{cases}$$

$$\varepsilon_t^* := \begin{cases} \frac{1}{2}\left(\frac{2^d-1}{2^{d+s\gamma_t-1}-1}\right)^{1/s} d^{1/s+\gamma_t/2}C_t(2A)^{d/s+\gamma_t}\left(\frac{1}{2^{(d-1)/s+\gamma_t}}\right)^{m_t+l}, & 0 < s < 1, \\ \frac{2^d-1}{2^{d/s+\gamma_t}-2}d^{1+\gamma_t/2}C_t(2A)^{d/s+\gamma_t}\left(\frac{1}{2^{d/s+\gamma_t-1}}\right)^{m_t+l}, & s \geq 1. \end{cases}$$

  (b) Assume that $d \in \mathbb{N}$ and f_t is differentiable with $\|\nabla f_t(x)\|_2 \leq C_t$ for all $x \in \text{supp}(f_t)$ with $C_t > 0$ and for all $t \in [-T, T]^q$. Then the algorithm can be applied with m_t and ε_t^* as in (a) with $\gamma_t = 1$ for all $t \in [-T, T]^d$.

We conclude this section with its main result.

Theorem 3.22. *Assume that $s > 0$ and $f_t \in L^p([-A, A]^d) \cap L^s([-A, A]^d)$ for some $p > 1$. Let X be an infinitely divisible random field and the control measure λ of the infinitely divisible random measure be the Lebesgue measure. Let $\varepsilon > 0$. If $\tilde{f}_t^{(n)}$ is calculated using the algorithm mentioned above, then*

$$Err_s(X(t), \tilde{X}^{(n)}(t)) \leq \varepsilon, \quad \forall t \in [-T, T]^q.$$

4. Simulation study

For the simulation study, we used two different types of kernel functions for α-stable random fields of dimension $d = 2$. The first one is an Epanechnikov-type kernel function defined by

(4.1) $$f_t(x) = b \cdot (a^2 - \|x - t\|_2^2)\mathbb{1}_{\{\|x-t\|_2 \leq a\}},$$

where $a > 0$ and $b > 0$, whereas for the second one, we take

(4.2) $$f_{(t_1,t_2)}(x_1, x_2) = b(a - |x_1 - t_1|)(a - |x_2 - t_2|)\mathbb{1}_{\{|x_1-t_1|\leq a, \ |x_2-t_2|\leq a\}}(x_1, x_2)$$

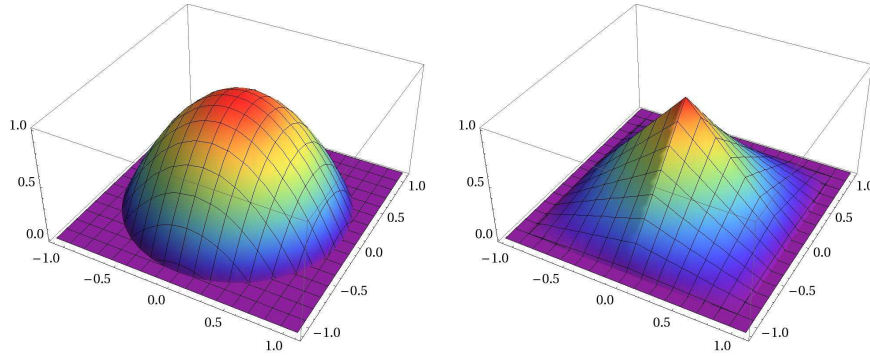where $a > 0$ and $b > 0$. Examples of both kernel functions are plotted in Figure 1.

FIGURE 1. The Epanechnikov-type kernel function (4.1) (left) and the kernel function (4.2) (right)

The main difference between these two types of kernel functions is that one can derive a simple formula for the integral of (4.2) over squares, but not for the integral

of (4.1). This does not affect the step function approach since the kernel functions are only evaluated there at the points ξ_k, but it does affect the wavelet approach because the input vector consists of such integrals of (4.1) and (4.2) over squares. Therefore, we have to expect a loss in computational performance for kernel (4.1) with the wavelet approach in this case.

Both functions (4.1) and (4.2) are Hölder-continuous with $(C_1, \gamma_1) = (2ab, 1)$ and $(C_2, \gamma_2) = (2\sqrt{2}ab, 1)$, respectively. We fix the parameters $\alpha = 1.5$, $\beta = 0$ and $[-T, T]^2 = [-1, 1]^2$ for both types of kernel functions.

For the remaining parameters, we started with the following configuration: $b = 1$, $a = 0.2$ and $\varepsilon = 0.5$. Furthermore, we divided $[-1, 1]^2$ into an equidistant grid of 100×100 points and chose $l = 0$ for the number of detail levels to be increased. Two realizations of the 1.5-stable random field X with kernels (4.1) and (4.2) are shown in Figure 2.
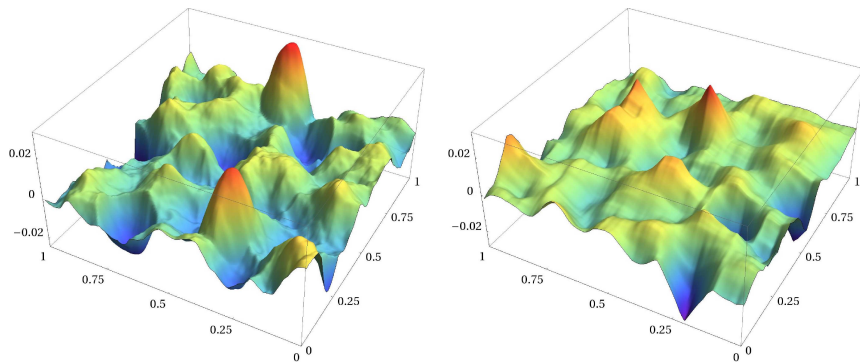

FIGURE 2. Realization of a symmetric 1.5-stable random field with kernel (4.1) (left) and kernel (4.2) (right)

First, we kept all parameters fixed and determined the computational time depending on the number of realizations. For the step function approach, each realization needs the same computational time. For the wavelet approach, however, the wavelet coefficients only have to be calculated for the first realization and can be stored afterwards. Therefore, any further realization needs less computational time. Table 1 shows the results for both the Epanechnikov-type kernel (4.1) and kernel (4.2). By trial-and-error, we figured out that a combination of $\varepsilon = \varepsilon_1 + \varepsilon_2 = 0.49 + 0.01$ performs quite good for the corresponding parameters in the wavelet algorithm in this case.

TABLE 1. Computational time in msec (Pentium(R) Dual-Core CPU E5400, 2.70GHz)) for the first and further realizations

	Kernel (4.1)	Kernel (4.2)
Step function approach	5.72	6.86
Wavelet approach (first realization)	70552.02	11546.07
Wavelet approach (further realizations)	1453.18	252.3

Second, we focused on kernel (4.2) and the computational time of any further realization except the first one and varied subsequently one of the parameters α, m (the number of pixels per row) and ε while all the other parameters were kept fixed. It turned out that the computational time decreased for the wavelet approach and increased for the step function approach when $\alpha \in [1,2]$ was decreased. For decreasing $\alpha \in (0,1]$, the computational time of both approaches increased. However, the step function approach performed at least twice as good as the wavelet approach. For decreasing ε, the computational time for the step function approach increased much faster than the one for the wavelet approach. Varying m affected the computational time of both approaches in a similar manner. We emphasize that we found parameter configurations where the wavelet approach performed better than the step function approach, e. g. $\alpha = 0.5$, $\varepsilon = 0.2$ and $m = 150$. The above results imply that neither of the two approaches outperforms the other one. For some combinations of the parameters, the step function approach was faster than the wavelet approach, for others it was slower.

Finally, we increased the parameter l successively for a field with 10×10 pixels while all other parameters were kept fixed and investigated the computational time for the wavelet approach for any further realization except the first one. Table 2 shows the corresponding results.

TABLE 2. Computational time in msec (Pentium(R) Dual-Core CPU E5400, 2.70GHz)) for different values of l (kernel (4.2))

l	0	1	2	3
Computational time	58.6	221.84	871.12	3252.32

One might have expected that the computational time tends to decrease if l is increased since the wavelet series usually consists of less summands when keeping the same level of precision. At the same time, however, more stable random variable simulations have to be performed for the calculation of the integrals

$$\int_{[-A,A]^d} \Psi^e_{j-2^{m+l},m+l} M(dx).$$

That is why for larger values of l, the computational time increases sharply.

A 2-stable random field is a Gaussian random field. Therefore, we can use the simulation methods to simulate such fields, too. The kernel function

$$f_t(x) = b \exp\{-\|x - t\|_2^2 / a\}, t \in \mathbb{R}^2,$$

corresponds to the isotropic and stationary covariance function

(4.3) $$C(h) \quad := \quad \text{Cov}(X(0,0), X(h,0)) = \pi a b^2 e^{-\frac{h^2}{2a}}.$$

We now want to investigate how our simulation methods perform compared with the circulant embedding method (see [25]) to simulate Gaussian random fields with the given covariance function (4.3) for $a = 0.05$, $b = 1$ and $[-T,T]^2 = [-0.5, 0.5]^2$ for a grid of size 100×100. We choose the circulant embedding method since it is exact in principle, and if exact simulation takes too much computational time, approximation techniques exist so that at least the one-dimensional marginal distributions are exact in principle.

For the comparison of the circulant embedding me-thod with the step function approach and the wavelet approach, we simulate 1000 fields and estimate their mean and their covariance function for the distances $0, 0.01, 0.02, \ldots, 0.5$. We then compare the values with the theoretical ones and require that the estimated values do not differ from the theoretical ones more than 0.01. If at least one value differs more than 0.01, the precision level is increased. The following table shows the computational time for each of the three methods.

TABLE 3. Computational time in msec (Pentium(R) Dual-Core CPU E5400, 2.70GHz)) for the circulant embedding method, the step function approach and the wavelet approach

	Circulant embedding	Step function approach	Wavelet approach
Computational time	48.43	9588.29	N/A

As one can see, the step function approach takes much more computational time than the circulant embedding method. For the wavelet approach, it was not possible to obtain the desired accuracy since the stored wavelet coefficients exceeded the available random-ac-cess memory (6 GB RAM). Here, the best approximation yielded an estimated mean and variance which differed from the corresponding theoretical quantities by 0.005 and 0.031, respectively, while only 9% of the estimated covariance function values where within the deviation allowance of 0.01. For Gaussian random fields, it is therefore advisable to use existing simulation methods such as the circulant embedding method that exploit the specific structure of these fields.

5. SUMMARY

We presented two approaches to simulate infinitely divisible random fields that are based on approximating the kernel function by a step function and by a wavelet series. For both approaches, we derived estimates for the approximation error $Err_s(X(t), \tilde{X}^{(n)}(t))$. In the simulation study we have seen that neither of the approaches outperformed the other one.

Let us compare the rates of convergence of the step function approach and the wavelet approach more generally. If f_t is Hölder-continuous, the error estimate for the step function approach is

$$(5.1) \qquad Err_s(X(t), \tilde{X}^{(n)}(t)) \leq C_1(d, C_t, \gamma_t, s, A) \left(\frac{1}{n}\right)^{\gamma_t}$$

for a constant $C_1(d, C_t, \gamma_t, s, A) > 0$. In the simulation study, we have seen that increasing the parameter l in order to get closer to the best n-term approximation is not so advantageous. Therefore, we consider the rate of convergence for the cut wavelet series with error estimate

$$(5.2) \qquad Err_s(X(t), \tilde{X}^{(n)}(t)) \leq C_2(d, C_t, \gamma_t, s, A) \left(\frac{1}{2^{\frac{d}{s}+\gamma_t}}\right)^{n}$$

with a constant $C_2(d, C_t, \gamma_t, s, A) > 0$. Notice that we cannot compare the error estimates directly because for the step function approach, n determines the number of cubes $((2n)^d)$ that form a partition of $[-A, A]^d$, while for the wavelet approach, n is the detail level. Therefore, we express the error bounds in terms of the number of

summands of the step function approximation (3.4) and the wavelet approximation (3.19), respectively, see Table 4.

TABLE 4. Number of summands of (3.4) and (3.19) and error bounds for (5.1) and (5.2) in terms of the number of summands

	Step function approach	Wavelet approach
Number of summands	$u = (2n)^d$	$u = 1 + (2^d - 1)d(2^{n+1} - 1)$
Error bounds	$O\left(\left(\frac{1}{u}\right)^{\gamma_t/d}\right)$	$O\left(\left(\frac{1}{u}\right)^{(d/s+\gamma_t)\ln 2}\right)$

Let us consider the number of random variables that need to be simulated for a single realization of a random field. For the step function approach, we need to simulate $(2n)^d$ random variables. For the wavelet approach, the number of random variables to simulate is equal to the number of cubes that form a partition of $[-A, A]^d$ in the finest detail level n: $2^{d(n+1)}$. In the examples of the simulation study, the number of random variables to be simulated for the step function approach increased much faster than for the wavelet approach when the value of ε was decreased.

Some remarks about the wavelet approach are in order. First, we have seen that one drawback is that the computation of the input vector for the fast wavelet transform may take quite a long time if no explicit formula for $\int_C f_t(x)dx$ is known, where C is a cube in $\mathbb{R}^d$. Second, for an arbitrary (not necessarily Haar) wavelet basis $\{\Psi_i\}_{i \in I}$, we would have to calculate $\int_C f_t(x)\Psi_i(x)dx$ for many i which mostly requires additional numerical integration. Interpolating wavelets can remedy this disadvantage since for this kind of wavelets bases, the wavelet coefficients are computed by evaluating the kernel function at a certain point. However, the interpolating wavelets themselves are no step functions any more, hence the simulation of the integrals $\int_{[-A,A]^d} \Psi_i(x)\Lambda(dx)$, where Ψ_i is an interpolating wavelet function, is much more complicated than for the Haar basis. Third, one could use adaptive wavelet methods in order to calculate the wavelet coefficients. This might decrease the computational time for the first random field realization. However, we have seen in the simulation study that increasing the parameter l has little advantage over the cut wavelet series ($l = 0$) since the negative effect of the increasing detail level and thus the need of more random variable simulations dominates the positive one of less summands in the wavelet series.

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