# The Brownian Motion

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## Definition 01: Stochastic process

Given a probability space $(\Omega, \mathcal{F}, P)$ and a measurable space $(E, \Sigma)$, an E-valued **stochastic process** is a family of E-valued random variables $X_t : \Omega \to E$ , indexed by an arbitrary set T (called the index set). That is, a stochastic process X is a family $\{X_t : t \in T\}$ where each $X_t$ is an E-valued random variable on $\Omega$. The space E is then called the **state space** of the process. When T$= \mathbb{N}$ (or T$= \mathbb{N}_0$) or any other countable set, $\{X_t\}$ is said to be a **discrete-time** process, and when T$= [0, \infty)$, it is called a **continuous-time** process. From now on: $T = [0, \infty)$ and $E = \mathbb{R}^n$.

## Definition 02: Trajectory

The function (defined on the index set $T = [0, \infty)$ and taking values in $\mathbb{R}^n$): $t \rightarrow X_t(\omega)$ is called the **trajectory** (or the sample path) of the stochastic process X corresponding to the outcome $\omega$. So, to every outcome $\omega \in \Omega$ corresponds a trajectory of the process which is a function defined on the index set $T$ and taking values in $\mathbb{R}^n$.

## Definition 03: (finite-dimensional) distributions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The **(finite-dimensional) distributions** of the process $\{X_t : t \in T\}$ are the measures $\mu_{t_1,\ldots,t_k}$ defined on $\mathbb{R}^{nk}$, k = 1, 2,... , by

$$\mu_{t_1,\ldots,t_k}(F_1 \times F_2 \times \ldots \times F_k) = P[X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k] \qquad t_i \in T; F_1, \ldots, F_k \in \mathcal{B}(\mathbb{R}^n)$$

## Definition 04: Modification

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T = [0, \infty)$ index set and $\{X_t : t \in T\}$, $\{Y_t : t \in T\}$ stochastic processes on $(\Omega, \mathcal{F}, P)$. Then we say that $X_t$ is a **version** (or a modification) of $Y_t$, if

$$P(\{\omega \in \Omega; X_t(\omega) = Y_t(\omega)\}) = 1 \qquad \forall t \in [0, \infty)$$

Note that if $X_t$ is a version of $Y_t$, then $X_t$ and $Y_t$ have the same finite-dimensional distributions.

### Definition 05: stationary processes

A stochastic process $X_t$ is called **stationary** if $X_{t_1}, \ldots, X_{t_n}$ have the same distribution for any $t_1, \ldots t_n \in T$.

## Remark 01: Kolmogorov's extension theorem

For all permutations $\sigma, t_1, ..., t_k \in T, k \in \mathbb{N}$ let $\nu_{t_1,\ldots,t_k}$ be probability measures on $\mathbb{R}^{nk}$ s.t.

$$\text{(K1) } \nu_{t_{\sigma(1)},\ldots,t_{\sigma(k)}}(F_1 \times ... \times F_k) = \nu_{t_1,\ldots,t_k}(F_{\sigma^{-1}(1)} \times ... \times F_{\sigma^{-1}(k)})$$

$$\text{(K2) } \nu_{t_1,\ldots,t_k}(F_1 \times ... \times F_k) = \nu_{t_1,\ldots,t_k,t_{k+1},\ldots,t_{k+m}}(F_1 \times ... \times F_k \times \mathbb{R}^n \times ... \times \mathbb{R}^n) \forall m \in \mathbb{N}$$

Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{X_t\}$ on $\Omega$, $X_t : \Omega \to \mathbb{R}^n$, s.t.

$$\nu_{t_1,\ldots,t_k}(F_1 \times ... \times F_k) = P[X_{t_1} \in F_1, ..., X_{t_k} \in F_k]$$

$\forall t_i \in T, k \in \mathbb{N}$ and all Borel Sets $F_i$

## Construction 1

To construct the stochastic process Brownian Motion it suffices, by the Kolmogorov extension theorem, to specify a family $\{\nu_{t_1,\dots,t_k}\}$ of probability measures satisfying (K1) and (K2).
Fix $x \in \mathbb{R}^n$ and define:

$$p(t,x,y) = \frac{1}{(2\pi t)^{\frac{n}{2}}} \cdot \exp\left(-\frac{||x-y||^2}{2t}\right) \qquad y \in \mathbb{R}^n, t > 0$$

, which is the probability density function of the normal distribution.

## Construction 2

If $0 \leq t_1 \leq \ldots \leq t_k$ define a measure (compare measure theory) $\nu_{t_1,\ldots,t_k}$ on $\mathbb{R}^{nk}$ by

$$\nu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) =$$

$$\int\limits_{F_1 \times \ldots \times F_k} p(t_1, x, x_1) p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k) dx_1 \cdots dx_k \qquad (1)$$

w.r.t. Lebesgue measure and the convention $\int\limits_{\mathbb{R}^n} p(0, x, y) dy = \delta_x(y)$.

## Construction 3

$\{\nu_{t_1,\ldots,t_k}\}$ satisfies (K1) and (K2), since we extend the defintion to all finite sequences of $t_i$'s by using (K1) and $\int\limits_{\mathbb{R}^n} p(t,x,y)dy = 1, \forall t \geq 0$. So by Kolmogorov's theorem there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{X_t\}_{t\geq 0}$ on $\Omega$ so that the finite-dimensional distributions are given by (1), s.t.

$$P(B_{t_1} \in F_1 \times \ldots \times B_{t_k} \in F_k) =$$

$$\int\limits_{F_1\times\ldots\times F_k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_k-t_{k-1},x_{k-1},x_k)dx_1\cdots dx_k \qquad (2)$$

### Definition 06: Brownian Motion

Such a process is called (a version of) **Brownian Motion** starting at x $\in \mathbb{R}^n$, w.l.o.g. $x = 0$.

## Remark 02: properties of Brownian Motion

(i) $P[(B_0 = x)] = 1$

(ii) Brownian Motion thus defined ist not unique. There exist several Quadruples $(B_t, \Omega, \mathcal{F}, P)$ such that (2) holds.

(iii) If $B_t = (B_t^{(1)}, \dots, B_t^{(n)})$ is n-dimensional Brownian Motion, then the 1-dimensional processes $\{B_t^{(j)}\}_{t \geq 0}$ are independent, 1-dimensional Brownian Motions.

(iv) $\mathrm{Cov}(B_t, B_s) = min\{s, t\}$ for one-dimensional Brownian Motions. For n-dimensional Brownian Motion: $\mathrm{Cov}(B_t, B_s) = min\{s, t\} \cdot E_n$

(v) The Brownian Motion is also called Wiener Process.

### Definition 06: (almost sure) continuous stochastic processes

A stochastic process $\{X_t : t \in T\}$ is **almost surely continuous**, if

$$P\left(\lim_{t \to s} X_t = X_s \forall s \in T\right) = 1.$$

## Remark 03: different Definition of Brownian Motion

In the literature we often find a different Definition of the Brownian Motion: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_t$ a stochastic process on $(\Omega, \mathcal{F}, P)$ $(T = [0, \infty))$. Then $X_t$ is a Brownian Motion if it satisfies the following four conditions:

1. $\{X_t : t \in T\}$ has independent increments (for any $0 \leq t_1 < t_2 < \cdots < t_n : X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent).
2. $X_{t_2} - X_{t_1} \sim N(0, t_2 - t_1)$ for any $t_1, t_2 \in T$ with $t_1 < t_2$
3. $X_0 = 0$ almost surely
4. the trajectories $t \to X_t(\omega), t \in T$, are continuous for any $\omega \in \Omega$.

Question: Does our construction from above also satisfies this definition?
Answer: Yes, it does!

## Proposition 01: our construction is a Brownian Motion

Our construction of the Brownian Motion satisfies 1. 2. and 3. from the definition above.

## Proof 1:

1. Let $Y \sim N(\mu, K)$ be an n-dimensional gaussian random vector and $A$ be a $(n \times n)$-Matrix. Then $AY \sim N(A\mu, AKA^T)$. This is a result from the explicit from of the characteristic function of $Y$. Now $k \in \mathbb{N}$ and:

$$
\begin{aligned}
0 &= t_0 \leq t_1 < t_2 < \cdots < t_k, \\
Y &= (X_{t_0}, \ldots, X_{t_k})^T, \\
Z &= (X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_k} - X_{t_{k-1}})^T, \\
A &= \begin{pmatrix} 1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ -1 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix}
\end{aligned}
$$

so that $Z = AY$.

## Proof 2:

Then $Z$ is also gaussian with a covariance matrix which is diagonal. Indeed: $\operatorname{cov}\left(X_{(t_{i+1})} - X_{(t_i)}, X_{(t_{j+1})} - X_{(t_j)}\right) = \min\{t_{i+1}, t_{j+1}\} - \min\{t_{i+1}, t_j\} - \min\{t_i, t_{j+1}\} + \min\{t_i, t_j\} = 0$ for $i \neq j$. Therefore the coordinates of $Z$ are uncorrelated. Because Z is gaussian distributed, the coordinates are independent and the increments of $X_t$ are independent too.

## Proof:

2. Let $0 \leq s < t$. Then $X_t - X_s \sim N(0, (t-s))$, because $Z = AY$ is gaussian distributed and $E(X_t) - E(X_s) = 0$ and $\mathrm{var}(X_t - X_s) = \mathrm{var}(X_t) - 2\,\mathrm{cov}(X_s, X_t) + \mathrm{var}(X_s) = t - 2\min\{s, t\} + s = t - s$.
3. Since $X_t \sim N(0, t) \Rightarrow X_0 \sim N(0, 0) \Rightarrow X_0 = 0$ almost surely.
4. We need one more Proposition for this result.

## Remark 04: Kolmogorov's continuity theorem

Suppose that the process $\{X_t\}_{t\geq 0}$ satisfies the following condition: For all $T > 0$ there exist positive constants $\alpha, \beta, D$ s.t.

$$E[||X_t - X_s||^{\alpha}] \leq D \cdot ||t - s||^{1+\beta} \qquad 0 \leq s, t \leq T \tag{3}$$

Then there exists a continuous version of X.

## Proposition 02: continuous version of Brownian Motion

Brownian Motion satisfies Kolmogorov's condition (3) with $\alpha = 4$, $D = n(n+2)$ and $\beta = 1$, and therefore the Brownian Motion has a continuous version.

## Proof:

Remember, if

$$Z \sim N(0, \sigma^2) \Rightarrow E(Z^k) = \mu_k = \begin{cases} 0 & \text{if } k \text{ odd} \\ (k-1)!! \cdot \sigma^k & \text{if } k \text{ even} \end{cases} \; k \in \mathbb{N},$$

$B_t^{(i)} \sim N(0, t) \forall i$ and $B_t^{(i)} - B_s^{(i)} \sim N(0, (t-s)) \forall i$. Then:

$$E[||B_t - B_s||^4] = \sum_{i=1}^{n} E[(B_t^{(i)} - B_s^{(i)})^4] + \sum_{j=1 j\neq i}^{n} E[(B_t^{(i)} - B_s^{(i)})^2 (B_t^{(j)} - B_s^{(j)})^2]$$

$$= n \cdot \frac{4!}{2! \cdot 4} \cdot (t-s)^2 + n(n-1)(t-s)^2 = n(n+2)(t-s)$$

□

We also notice that the Brownian Motion has a continuous version and therefore our construction of the Brownian Motion satisfies the definition from above.

## Example 01: continuity properties of Trajectory

In general the (finite-dimensional) distribution alone does not give all the information regarding to continuity properties of stochastic processes. To illustrate that, consider the following example:

## Proof:

Let $(\Omega, \mathcal{F}, P) = ([0,\infty], \mathcal{B}, \mu)$ where $\mathcal{B}$ denotes the Borel $\sigma$-Algebra on $[0,\infty)$ be a probability space. $\mu$ is a probability measure on $[0,\infty)$, with no single point mass $(\nexists x \in [0,\infty) : \mu(x) > 0)$. Define:

$$X_t(\omega) = \begin{cases} 1, & \text{if } t = \omega \\ 0, & \text{otherwise} \end{cases} \qquad \text{and } Y_t(\omega) = 0 \qquad \forall (t,\omega) \in [0,\infty) \times [0,\infty)$$

Let $t_i \in [0,\infty)$ and $F_1, ..., F_k \in \mathcal{B}([0,\infty))$ Then:

$$\mu[X_{t_1} \in F_1, ..., X_{t_k} \in F_k] = \mu(\omega \in [0,\infty) : X_{t_1}(\omega) \in F1, ..., X_{t_k}(\omega) \in F_k) =$$

$$\begin{cases} 1, & 0 \in F_k \forall k \\ 0, & \text{otherwise} \end{cases} = \mu(\omega \in [0,\infty) : Y_{t_1}(\omega) \in F1, ..., Y_{t_k}(\omega) \in F_k) =$$

$$\mu[Y_{t_1} \in F1, ..., Y_{t_k} \in F_k].$$

## Proof:

Notice that $\mu([0,\infty) \setminus \{t_1, \dots, t_k\}) = \mu([0,\infty)) = 1$. Therefore $X_t$ and $Y_t$ have the same distributions. Also $\mu(\omega \in [0,\infty) : X_t(\omega) = Y_t(\omega)) = 1 \Rightarrow X_t$ is a version of $Y_t$. And yet we have that $t \to Y_t(\omega)$ is continuous for all $\omega$, while $t \to X_t(\omega)$ is discontinuous for all $\omega$.

## Example 02: Sum of Brownian Motions

Let $B_t$ be Brownian Motion and fix $t_0 \geq 0$. Then $\tilde{B}_t := B_{t_0+t} - B_{t_0}; t \geq 0$ is a Brownian Motion.

## Proof:

We will show that $\tilde{B}_t$ satisfies the remark 03:

1. Since $B_t$ is a Brownian Motion, $B_t$ has independent increments. Therefore $\tilde{B}_t$ has independent increments too (Notice: $\tilde{B}_{t_1} - \tilde{B}_{t_2} = B_{t_0+t_2} - B_{t_0} - B_{t_0+t_1} + B_{t_0} = B_{t_0+t_2} - B_{t_0+t_1}$, which is independent of $B_{t_0+t_3} - B_{t_0+t_4} \forall t_1, t_2, t_3 \in T$).
2. $\tilde{B}_{t_2} - \tilde{B}_{t_1} \sim N(0, t_2 - t_1)$ for any $t_1, t_2 \in T$ with $t_1 < t_2$, because $\tilde{B}_{t_2} - \tilde{B}_{t_1} = B_{t_0+t_2} - B_{t_0+t_1} \sim N(0, t_2 - t_1)$
3. $\tilde{B}_0 := B_{t_0+0} - B_{t_0} = 0$ almost surely.
4. Since the trajectories $t \to B_t(\omega), t \in T$, are continuous for any $\omega \in \Omega$ $t \to \tilde{B}_t(\omega), t \in T$, are continuous too.

## Example 03: Brownian Motion has stationary increments

Let $(\mathcal{C}[0,\infty), \mathcal{B}(\mathcal{C}[0,\infty)), P)$ be the canonical probability space. The Brownian motion $B_t$ has stationary increments, i.e. that the process $\{B_{t+h} - B_t\}_{h\geq 0}$ has the same distribution for all $t$.

Proof: Since $\tilde{B}_t := B_{t_0+t} - B_{t_0}$ is a Brownian Motion, $B_{t_0+t} - B_{t_0} \sim N(0,t)$. $\square$

## Example 04:

Let $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P)$ be a probability space, $\{B_t : t \in [0, \infty)\}$ $n$-dimensional Brownian Motion on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P)$ and let $K \subset \mathbb{R}^n$ have zero $n$-dimensional Lebesgue measure. Then the expected total length of time that $B_t$ spends in $K$ is zero. Proof:

$$E\left(\int_0^\infty \chi_K(B_t)dt\right) = \int_0^\infty E(\chi_K(B_t))dt = \int_0^\infty P(B_t \in K)dt$$

$$= \int_0^\infty \frac{1}{(2\pi t)^{\frac{n}{2}}} \underbrace{\left(\int_K exp\left(-\frac{||x-y||^2}{2t}\right)dy\right)}_{\text{=0 (compare measure theory and } \chi_K \geq 0 \text{ a.s.)}} dt = 0$$

$\forall x \in \mathbb{R}^n, K \subset \mathbb{R}^n$ with K Lebesgue measure zero.

$\square$

## Definition 07: bounded variation

For $t > 0$ a function $f : [0, t] \to \mathbb{R}$ is said to be of **bounded variation**, if

$$V_f^{(1)} := \sup \left\{ \sum_{k=1}^{m} |f(t_k) - f(t_{k-1})| : m \in \mathbb{N}, 0 = t_0 < \cdots < t_m = t \right\}$$

is finite. Otherwise f is said to be of unbounded variation.

## Proposition 03: Brownian Motion is of unbounded variation

The Brownian Motion is almost surely of unbounded Variation (for all $t > 0$).
Proof: w.l.o.g. t=1 (for $\neq$ 1 use the scaling properties of the Brownian Motion).
Let $X_t$ be a Brownian Motion, then:

$$Z_n = \sum_{k=1}^{2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| = \sqrt{2^n} \underbrace{\frac{1}{2^n} \sum_{k=1}^{2^n} |\sqrt{2^n}(X_{k2^{-n}} - X_{(k-1)2^{-n}})|}_{\overset{n\to\infty}{\longrightarrow} E[|X_1|] \text{ in probability (law of large numbers)}}$$

The convergence in probability follows from the law of large numbers , because all summands are independent and distributed like $|X_1|$. Therefore $Z_n \to \infty$ in probability. Because of the triangle inequality the random variables $Z_n$ are monotonically increasing thus $Z_n$ converges almost surely to infinity. $\square$

## Proposition 04: Brownian Motion is of finite quadratic variation

Let $\pi_m : 0 = t_0^m < t_1^m < \cdots < t_m^m = t (m \in \mathbb{N})$ be a partition of $[0, t]$ which mesh size converges to zero. Then:

$$\lim_{m \to \infty} \sum_{k=1}^{m} \left( X_{t_k^m} - X_{t_{k-1}^m} \right)^2 = t, \text{ in probability.}$$

## Proof 1:

Define: $Z_m = \sum_{k=1}^{m} \left( X_{t_k^m} - X_{t_{k-1}^m} \right)^2$. Then:

$$E[Z_m] = \sum_{k=1}^{m} \underbrace{E\left[ \left( X_{t_k^m} - X_{t_{k-1}^m} \right)^2 \right]}_{=t_k^m - t_{k-1}^m} = t$$

and

## Proof 2:

$$\mathrm{var}(Z_m) = \sum_{k=1}^{m} \underbrace{\mathrm{var}\left(\left(X_{t_k^m} - X_{t_{k-1}^m}\right)^2\right)}_{=\left(t_k^m - t_{k-1}^m\right)^2 \mathrm{var}\left(X_1^2\right)}$$

$$\leq \sup_{k=1,\dots,m} \left(t_k^m - t_{k-1}^m\right) \sum_{k=1}^{m} \left(t_k^m - t_{k-1}^m\right) \ \mathrm{var}\left(X_1^2\right) = \ \mathrm{var}\left(X_1^2\right) t \sup_{k=1,\dots,m} \left(t_k^m - t_{k-1}^m\right)$$

$$\xrightarrow{m\to\infty} 0.$$

The convergence is a consequence of the Tschebyscheff inequality. □

## Remark 04:

- One can show: $\lim_{m \to \infty} \sum_{k=1}^{m} \left( X_{t_k^m} - X_{t_{k-1}^m} \right)^2 = t$, for almost all $\omega \in \Omega$.
- note that a process may be of finite quadratic variation and its paths be nonetheless almost surely of infinite quadratic variation for every $t > 0$ (e.g. Brownian Motion)

## Literature:

[1] Øksendal, B. (2010) *Stochastic Differential Equations: An Introduction with Applications*. Springer