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Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X : \Omega \rightarrow \mathbb{R}^n$ be a random variable, $\mathbb{E}[|X|] < \infty$ and $\mathcal{H} \subset \mathcal{F}$ a $\sigma$-algebra, then the conditional expectation of $X$ given $\mathcal{H}$, denoted by $\mathbb{E}[X|\mathcal{H}]$, is defined as follows:

**Definition:**
$\mathbb{E}[X|\mathcal{H}]$ is the (a.s. unique) function from $\Omega$ to $\mathbb{R}^n$ satisfying:

1. $\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable
2. $\int_{\mathcal{H}} \mathbb{E}[X|\mathcal{H}] \, d\mathbb{P} = \int_{\mathcal{H}} X \, d\mathbb{P}$ for all $H \in \mathcal{H}$
Example:
Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the set of numbers of a die, $\mathcal{P}(\Omega) = \mathcal{F}$ and $X : \Omega \to \mathbb{N}$ be the random variable with $X(\omega) = \omega$ (the number of the die). Now we hide the numbers 1 and 6 by covering them. Thus our observations get inaccurate and our new $\sigma$-algebra is $\mathcal{H} = \sigma(\{2\}, \{3\}, \{4\}, \{5\}, \{1, 6\})$. So $\mathcal{H} \subset \mathcal{F}$.

What is happening to $X$? $X$ is not measurable to $\mathcal{H}$. So we create an appropriate RV ($\mathbb{E}[X|\mathcal{H}]$) s.t.

1. $\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable
2. $\mathbb{E}[\mathbb{E}[X|\mathcal{H}] \cdot X_H] = \mathbb{E}[X \cdot X_H]$ for all $H \in \mathcal{H}$

We define

$$\mathbb{E}[X|\mathcal{H}](\omega) := X(\omega), \quad \text{for } \omega = 2, 3, 4, 5$$

$$\mathbb{E}[X|\mathcal{H}](\omega) := \frac{1+6}{2} = 3.5, \quad \text{for } \omega = 1, 6$$

Obviously $\mathbb{E}[X|\mathcal{H}]$ satisfies 1. and 2. .
Proof:
We want to show the existence and the a.s. uniqueness of \( \mathbb{E}[X|\mathcal{H}] \). Let \( \nu \) be the integral of \( X \) over a set \( H \):

\[
\nu(H) := \int_H X \, d\mathbb{P} \text{ for all } H \in \mathcal{H}
\]

It is easy to see, that \( \nu \) is a finite signed measure on \( \mathcal{H} \). Furthermore, it is \( \forall \, H \in \mathcal{H} \), if \( \mathbb{P}(H) = 0 \), then \( \nu(H) = 0 \). So \( \nu \) is absolutely continuous w.r.t. \( \mathbb{P}|\mathcal{H} \).

As \( (\Omega, \mathcal{H}, \mathbb{P}|\mathcal{H}) \) is a \( \sigma \)-finite space, we can apply the theorem of Radon-Nikodym, which says there is a \( \mathbb{P}|\mathcal{H} \)-unique \( \mathcal{H} \)-measurable function \( F \) on \( \Omega \) such that

\[
\nu(H) := \int_H F \, d\mathbb{P} \text{ for all } H \in \mathcal{H}
\]

We define \( \mathbb{E}[X|\mathcal{H}] := F \) and this function is unique w.r.t. to the measure \( \mathbb{P}|\mathcal{H} \).
Now we consider the most important properties of the conditional expectation:

Theorem:
Suppose $Y : \Omega \to \mathbb{R}^n$ is another random variable with $E[|Y|] < \infty$ and let $a, b \in \mathbb{R}$. Then

a) $E[aX + bY|\mathcal{H}] = aE[X|\mathcal{H}] + bE[Y|\mathcal{H}]$

b) $E[E[X|\mathcal{H}]] = E[X]$

c) $E[X|\mathcal{H}] = X$ if $X$ is $\mathcal{H}$ – measurable

d) $E[X|\mathcal{H}] = E[X]$ if $X$ is independent of $\mathcal{H}$

e) $E[Y \cdot X|\mathcal{H}] = Y \cdot E[X|\mathcal{H}]$ if $X, Y \in L^2$ and $Y$ is $\mathcal{H}$ – measurable, where $\cdot$ denotes the usual inner product in $\mathbb{R}^n$
Proof:
b) Assume $H = \Omega \in \mathcal{H}$. Then

$$
\mathbb{E}[\mathbb{E}[X|\mathcal{H}] \cdot \chi_H] = \int_H \mathbb{E}[X|\mathcal{H}] d\mathbb{P} \overset{2.}{=} \int_H X d\mathbb{P} = \mathbb{E}[X]
$$

c) As $X$ is $\mathcal{H}$-measurable, $X$ satisfies both 1. and 2. . Because of that, and the fact that $\mathbb{E}[X|\mathcal{H}]$ is a.s. unique, we conclude $X = \mathbb{E}[X|\mathcal{H}]$.

d) We show, that $\mathbb{E}[X]$ satisfies 1. and 2. . As $\mathbb{E}[X]$ is a constant, 1. is satisfied. If $X$ is independent of $\mathcal{H}$ we have for $H \in \mathcal{H}$

$$
\int_H \mathbb{E}[X] d\mathbb{P} = \mathbb{E}[X] \cdot \mathbb{P}[H] = \int_\Omega X d\mathbb{P} \cdot \int_\Omega \chi_H d\mathbb{P}
$$

$$
= \int_\Omega X \cdot \chi_H d\mathbb{P} = \int_H X d\mathbb{P}
$$
e) We show that $Y \cdot \mathbb{E}[X|\mathcal{H}]$ satisfies 1. and 2. As $Y$ and $\mathbb{E}[X|\mathcal{H}]$ are both measurable w.r.t. $\mathcal{H}$, we conclude that the product is also $\mathcal{H}$-measurable. To show property 2., we first consider $Y = \mathcal{X}_G$ ($\mathcal{H}$-measurable) for some $G \in \mathcal{H}$. Then for all $H \in \mathcal{H}$

$$\int_H Y \cdot \mathbb{E}[X|H]d\mathbb{P} = \int_{H \cap G} \mathbb{E}[X|H]d\mathbb{P} = \int_{H \cap G} X \ d\mathbb{P} = \int_{H} YX \ d\mathbb{P}$$

Similarly, we obtain that the result is true if

$$Y := \sum_{j=1}^{m} c_j \mathcal{X}_{G_j}, \text{ where } G_j \in \mathcal{H}.$$ 

As we can approximate every $\mathcal{H}$-measurable RV $Y$ by such simple functions, we proved the statement.
Theorem:
Let $\mathcal{G}, \mathcal{H}$ be $\sigma$-algebras such that $\mathcal{G} \subset \mathcal{H}$. Then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}].$$

Proof:
If $G \in \mathcal{G}$ then $G \in \mathcal{H}$ and therefore

$$\mathbb{E}[\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] \cdot \mathcal{X}_G] = \int_G \mathbb{E}[X|\mathcal{H}]d\mathbb{P} = \int_G X \ d\mathbb{P}$$

Once again, 1. and 2. are satisfied. Hence $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]$ by uniqueness.
**Theorem:** (The Jensen inequality)
If $\phi : \mathbb{R} \to \mathbb{R}$ is convex and $\mathbb{E}[|\phi(X)|] < \infty$ then

$$\phi(\mathbb{E}[X|\mathcal{H}]) \leq \mathbb{E}[\phi(X)|\mathcal{H}]$$

**Corollary:**
(i) $|\mathbb{E}[X|\mathcal{H}]| \leq \mathbb{E}[|X||\mathcal{H}]$
(ii) $|\mathbb{E}[X|\mathcal{H}]|^2 \leq \mathbb{E}[|X|^2|\mathcal{H}]$

**Proof:**
(i) It is

$$|\mathbb{E}[X|\mathcal{H}]| = |\mathbb{E}[X^+ - X^-|\mathcal{H}]| = |\mathbb{E}[X^+|\mathcal{H}] - \mathbb{E}[X^-|\mathcal{H}]| \leq \mathbb{E}[X^+|\mathcal{H}] + \mathbb{E}[X^-|\mathcal{H}] = \mathbb{E}[|X||\mathcal{H}]$$

(ii) Define $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(x) := x^2$. Then $\phi$ is convex and we can apply the Jensen inequality on $\phi(\mathbb{E}[X|\mathcal{H}])$. 
Corollary:
If \( X_n \to X \) in \( L^2 \) then \( \mathbb{E}[X_n|\mathcal{H}] \to \mathbb{E}[X|\mathcal{H}] \) in \( L^2 \).

Proof:
We have to show:
(1) \( \mathbb{E}[X|\mathcal{H}], \mathbb{E}[X_n|\mathcal{H}] \in L^2 \ \forall \ n \)
(2) \( \lim_{n \to \infty} \mathbb{E}[(\mathbb{E}[X_n|\mathcal{H}] - \mathbb{E}[X|\mathcal{H}])^2] = 0 \)

It is

\[
\mathbb{E}[(\mathbb{E}[X_n|\mathcal{H}])^2] = \mathbb{E}[|\mathbb{E}[X_n|\mathcal{H}]|^2] \leq \mathbb{E}[\mathbb{E}[|X_n|^2|\mathcal{H}]] = \mathbb{E}[\mathbb{E}[X_n^2|\mathcal{H}]] = \mathbb{E}[X_n^2] < \infty
\]

So \( \mathbb{E}[X_n|\mathcal{H}] \in L^2 \ \forall \ n \). Similarly we obtain that \( \mathbb{E}[X|\mathcal{H}] \in L^2 \).
To show (2), we first take a look on

$$\mathbb{E}[(\mathbb{E}[X_n | \mathcal{H}] - \mathbb{E}[X | \mathcal{H}])^2] = \mathbb{E}[(\mathbb{E}[X_n - X | \mathcal{H}])^2] \leq \mathbb{E}[\mathbb{E}[(X_n - X)^2 | \mathcal{H}]] = \mathbb{E}[(X_n - X)^2]$$

As \( n \) was arbitrary, we conclude:

$$\lim_{n \to \infty} \mathbb{E}[(\mathbb{E}[X_n | \mathcal{H}] - \mathbb{E}[X | \mathcal{H}])^2] \leq \lim_{n \to \infty} \mathbb{E}[(X_n - X)^2] = 0$$

It is \( \mathbb{E}[(\mathbb{E}[X_n | \mathcal{H}] - \mathbb{E}[X | \mathcal{H}])^2] \geq 0 \) and we follow

$$\lim_{n \to \infty} \mathbb{E}[(\mathbb{E}[X_n | \mathcal{H}] - \mathbb{E}[X | \mathcal{H}])^2] = 0$$
Martingales

Let \((\Omega, \mathcal{N}, \mathbb{P})\) be a probability space and let \(\{\mathcal{N}_t\}_{t \geq 0} \subset \mathcal{N}\) be a filtration, i.e. \(\{\mathcal{N}_t\}_{t \geq 0}\) is a family of increasing \(\sigma\)-algebras.

**Definition:**
A stochastic process \(\{N_t : t \geq 0\}\) with \(N_t : \Omega \to \mathbb{R}\) is adapted if \(N_t\) is \(\mathcal{N}_t\)-measurable \(\forall t \geq 0\).

**Definition:**
A stochastic process \(\{N_t : t \geq 0\}\) is a martingale if

1. \(N_t\) is \(\mathcal{N}_t\)-adapted
2. \(\mathbb{E}[|N_t|] < \infty\)
3. \(\mathbb{E}[N_s | \mathcal{N}_t] = N_t\ \forall 0 \leq t \leq s\)
**Definition:**
\( \{N_t : t \geq 0\} \) is called submartingale if 1. and 2. and 3(a).
\[ \mathbb{E}[N_s | N_t] \geq N_t \quad \forall 0 \leq t < s \]

X is called supermartingale if 1. and 2. and 3(b).
\[ \mathbb{E}[N_s | N_t] \leq N_t \quad \forall 0 \leq t < s \]

**Example:**
Brownian Motion \( \{B(t) : t \geq 0\} \) is a martingale w.r.t. to the natural filtration \( \mathcal{F}_t = \sigma \{B(s) : 0 \leq s \leq t\} \).

1. follows by definition and 2. is satisfied as \( \mathbb{E}[B(t)] = 0 \) for all \( t \geq 0 \). Furthermore, we have
\[ \mathbb{E}[B(t) | \mathcal{F}_s] = \mathbb{E}[B(s) + (B(t) - B(s)) | \mathcal{F}_s] = \mathbb{E}[B(s) | \mathcal{F}_s] + \mathbb{E}[(B(t) - B(s)) | \mathcal{F}_s] = B(s) + \mathbb{E}[B(t) - B(s)] = B(s) \]

independent \( N(0, t-s) \)
Example:
Assume \( \{ N_t : t \geq 0 \} \) is a martingale and \( \{|N_t| : t \geq 0\} \in L^1 \).
Then \( \{|N_t| : t \geq 0\} \) is a submartingale.
Proof: Obviously 1. and 2. are satisfied. To show 3(a)., we use the Jensen inequality.

\[
\mathbb{E}[|N_s||N_t|] \geq |\mathbb{E}[N_s|N_t]| = |N_t| \quad \forall 0 \leq t < s
\]

In conclusion, we obtain that a convex function of a martingale is a submartingale.

Example:
We consider \( t \in \mathbb{N}_0 \) (discrete time). A gambler wins 1$ when a coin comes up heads and loses 1$ when the coin comes up tails. Suppose now that the coin comes up heads with probability \( p \leq \frac{1}{2} \). On average, the gambler loses money and his fortune over time is a supermartingale.
As in customary we will assume that each $\mathcal{N}_t$ contains all the null sets of $\mathcal{N}$, that $t \to N_t(\omega)$ is right continuous for a.a. $\omega$ and that $\{\mathcal{N}_t\}$ is right continuous, in the sense that $\mathcal{N}_t = \bigcap_{s > t} \mathcal{N}_s$ for all $t \geq 0$.

**Theorem:** (Doob’s Martingale convergence theorem 1)

Let $N_t$ be a right continuous supermartingale, $\sup_{t > 0} \mathbb{E}[N_t^-] < \infty$ with $N_t^- = \max(-N_t, 0)$. Then

$$N(\omega) = \lim_{t \to \infty} N_t(\omega)$$

exists for a.a. $\omega$ and $\mathbb{E}[N^-] < \infty$.

If we assume that $N_t(\omega)$ is bounded in $L^1$ for all $t$, then $N(\omega)$ is finite a.s.
Example:
Let us take a look on the harmonic series: We know that
\[
\sum_{k=1}^{\infty} \frac{1}{k} = \infty
\]

But what about the following series
\[
\sum_{k=1}^{\infty} \frac{\xi_k}{k}
\]

where \(\xi_k\) are independent and identically distributed RVs with
\[
P[\xi_k = +1] = P[\xi_k = -1] = \frac{1}{2}
\]

If we consider the case \(\xi_1 = -1, \xi_2 = +1, \xi_3 = -1, \ldots\) we have
\[
\sum_{k=1}^{\infty} \frac{\xi_k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \ldots < \infty
\]
because of the alternating series test.
So we assume: \( \sum_{k=1}^{\infty} \frac{\xi_k}{k} < \infty \).

**Proof:**

Consider the partial sum \( S_n := \sum_{k=1}^{n} \frac{\xi_k}{k} \) with \( S_0 := 0 \).

\( S_n \) is a martingale w.r.t. \( \sigma(\xi_1, \ldots, \xi_n) \):

1. \( S_n \) is \( \sigma(\xi_1, \ldots, \xi_n) =: \mathcal{F}_n \) adapted

3. \( \mathbb{E}[S_n|\mathcal{F}_{n-1}] = \mathbb{E}[S_{n-1} + \xi_n|\mathcal{F}_{n-1}] = \mathbb{E}[S_{n-1}|\mathcal{F}_{n-1}] + \mathbb{E}[\xi_n|\mathcal{F}_{n-1}] \)

   \[ = S_{n-1} + \mathbb{E}[\xi_n] \mathbb{E}[\xi_k] = 0 \]

   \[ = S_{n-1} \]

For property 2., we need to show \( \mathbb{E}[|S_n|] < \infty \) \( \forall n \in \mathbb{N} \).

Therefore we prove that \( S_n \) is bounded in \( L^2 \). Then \( S_n \) is bounded in \( L^1 \) and property 2. is fulfilled.
Consider:

$$\mathbb{E}[S_n^2] = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2 = \text{Var}(S_n) = \sum_{k=1}^{n} \text{Var}\left[\xi_k\right]$$

$$\text{Var}\left[\xi_k\right] = 1 \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6} \quad \forall n \in \mathbb{N}$$

As $$\lim_{n \to \infty} \mathbb{E}[S_n^2] = \frac{\pi^2}{6}$$, it also follows that $$\sup_{n \in \mathbb{N}} \mathbb{E}[S_n^2] \leq \frac{\pi^2}{6}$$. So $$\{S_n\}_{n \in \mathbb{N}}$$ is bounded in $$L^2$$

$$\Rightarrow \{S_n\}_{n \in \mathbb{N}}$$ is bounded in $$L^1$$

We conclude that 2. property is fullfilled. So $$S_n$$ is a martingale and thus also a supermartingale.
Because of the fact, that \( \{S_n\}_{n \in \mathbb{N}} \) is bounded in \( L^1 \), we can apply Doob’s martingale convergence theorem \( \Rightarrow \lim_{n \to \infty} S_n \) exists in \( \mathbb{R} \) a.s.

Notice that if \( \xi_k \) are positive RV with \( \text{Var}(\xi_k) = \sigma \) for all \( k \in \mathbb{N} \), we can achieve the same result.

**Definition:**
A family \( C \) of RV \( N_t \) on a probability space is uniformly integrable (UI) if

\[
\lim_{K \to \infty} \left( \sup_{t \geq 0} \mathbb{E}[|N_t| \mathbb{1}_{|N_t| > K}] \right) = 0
\]

**Theorem:**
If a family \( C \) is bounded in \( L^p(p > 1) \), then it is UI.
**Theorem:** (Doob’s martingale convergence theorem 2)

Let $N_t$ be a right-continuous supermartingale. Then the following are equivalent:

1. $\{N_t\}_{t \geq 0}$ is uniformly integrable

2. $\exists \text{ RV } N \in L_1 \text{ s.t. } N_t \overset{a.e.,L^1}{\rightarrow} N.$

**Example:**

We already know that $\lim_{n \to \infty} S_n(\omega) = S(\omega)$, where $S$ is a finite RV. As $\{S_n\}_{n \in \mathbb{N}}$ is $L^2$-bounded, it is UI. With Doob’s martingale convergence theorem 2 we also obtain that $\lim_{n \to \infty} \mathbb{E}[|S_n - S|] = 0.$
References

1. Seminar papers; Appendix B: Conditional Expectation, Appendix C: Uniform integrability and Martingale convergence


4. Zakhar Kabluchko: Stochastic Processes (Stochastik 2), 29.05.2014
Thank you for your attention