Stochastic Simulation

Problem Sheet 5

Deadline: May 28, 2015 at noon before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (3 points)

Let $X$, $Y$ and $Z$ be three discrete random variables with values in $E$, $F$ and $G$, respectively. Prove the following statement:

If there is a function $g: E \times F \rightarrow [0, 1]$ such that $P(X = x | Y = y, Z = z) = g(x, y)$ for all $x \in E, y \in F, z \in G$, then $P(X = x | Y = y) = g(x, y)$ for all $x \in E, y \in F$, and $X$ and $Z$ are conditionally independent given $Y$.

Exercise 2 (theory) (3 points)

Consider a Markov random field $X$ on a finite set $S$ with respect to the neighborhood system defined by the graph $G = (S, E)$. Let $G_A = (A, E_A)$ with $A \subset S$ and $E_A = \{(s_1, s_2) \in E : s_1, s_2 \in A\}$ be a subgraph of $G$. Show that if $E$ contains no edges between $A$ and $S \setminus A$, then $X_A = \{X_s, s \in A\}$ is a Markov random field on $G_A$.

Exercise 3 (theory) (3 + 2 points)

Consider a Markov random field $X$ on a finite set $S$ with respect to the neighborhood system defined by the graph $G = (S, E)$. Let $M \subset S$ be the set of all vertices without neighbors. Show that

a) The set $X_M = \{X_s, s \in M\}$ consists of independent random variables.

b) $X_M$ is independent of $X_{S \setminus M}$.

Exercise 4 (theory) (3 points)

Consider a Markov random field $X$ on a finite set $S$ with respect to the neighborhood system defined by the graph $G = (S, E)$. Let $\tilde{G} = (V, \tilde{E})$ be another graph with the same vertices as $G$ but different edges. Is $X$ a Markov random field with respect to the neighborhood system defined by $\tilde{G}$? Give a counter example or provide a proof of your answer.
Exercise 5 (programming) (3 + 2 points)

Consider the set $A$ of all $5 \times 5$ matrices with values in $\{0, 1\}$ and the subset $A \subset \Omega$ of all such matrices which have no 1s directly next to each other in a row or column, like in Exercise 10 on the last problem sheet. This time, we do not want to draw uniformly from $\Omega$ but according to the distribution

$$\pi_T(x) = \frac{1}{Z_T} e^{-\frac{1}{T}E(x)} \quad \forall x \in A$$

with the energy function

$$E(x) = \begin{cases} -\sum_{s \in S} x_s \log(\lambda) & \text{if } x \in \Omega, \\ \infty & \text{otherwise}, \end{cases}$$

where $T = 1$ and $\lambda \in (0, \infty)$. Here $S$ denotes the set of all sites in the matrix $x$.

a) Write a Matlab program to draw from $\pi$ approximately, using the Metropolis algorithm.

b) Run your program from a) for $\lambda = 0.5, 1$ and $1.5$ and at least $N = 10^5$ steps and estimate the expected number of ones in the resulting random matrix.

*Hint:* You can do so by averaging over all matrices in the Markov chain you run.

Exercise 6 (programming) (2 + 3 + 2 points)

Consider two Markov chains $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ with the following transition graph.

![Transition Graph](attachment:image.png)

a) Write out a random mapping representation of $\{X_n\}_{n \in \mathbb{N}}$ using Bernoulli distributed random variables.

b) Write a Matlab program to simulate the random mapping coupling of $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ for $p = 0.6$, $X_0 = 1$ and $Y_0 = 7$ using a) and plot a realization of it up to the coupling time $\tau_{\text{couple}} = \inf\{n \geq 0 : X_n = Y_n\}$.

c) Run your program from b) at least $N = 10^4$ times and estimate the expected coupling time, $E\tau_{\text{couple}}$. 