Dr. Tim Brereton
Lisa Handl

Summer Term 2015

Stochastic Simulation

Problem Sheet 7

Deadline: June 18, 2015 at noon before the exercises

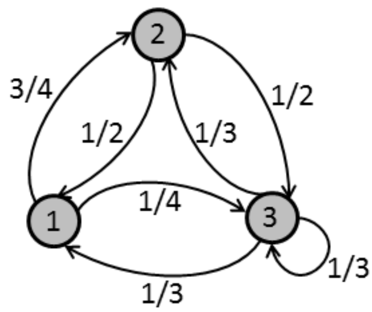
Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (4 points)

A professor has m umbrellas. He walks to the office in the morning and walks home in the evening. If it is raining he likes to carry an umbrella and if it is fine he does not. Suppose that it rains on each journey with probability $p \in (0,1)$, independently of past weather. What is the long-run proportion of journeys on which the professor gets wet?

Exercise 2 (theory) (4 + 2 points)

Consider a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with transition graph like in Exercise 4 on Problem Sheet 3, i.e.,



a) We can define a second Markov chain, $\{Y_n\}_{n\in\mathbb{N}}$, by $Y_n = (X_n, X_{n+1})$. Write out the state space and transition probabilities of $\{Y_n\}_{n\in\mathbb{N}}$ and calculate its stationary distribution.

   Hint: If you want, you can use a computer to solve for the stationary distribution (e.g., using `eig`). By hand is possible, too.

b) Calculate the average velocity with which the chain $\{X_n\}_{n\in\mathbb{N}}$ moves in clockwise direction on the graph above. This is defined as $\lim_{n\to\infty} \frac{d(n)}{n}$, where $d(n)$ is the distance traveled in the clockwise direction after n steps, i.e., $d(0) = 0$ and

$$d(n) = \begin{cases} d(n-1)+1, & \text{if the step from } X_{n-1} \text{ to } X_n \text{ is in the clockwise direction,} \\ d(n-1)-1, & \text{if it is in the counter-clockwise direction,} \\ d(n-1), & \text{otherwise.} \end{cases}$$

Exercise 3 (programming) (3 + 2 points)

Suppose you are a big soccer fan and each time your favourite team plays, you bet that they will win. In each match the players are either motivated or not and they can either have a strong opponent or a weak one, so there are 4 possible states. In each of these cases you win or lose something:

1. unmotivated players, weak opponent $\Rightarrow$ team wins closely (+5 €)
2. motivated players, weak opponent $\Rightarrow$ team wins (+10 €)
3. motivated players, strong opponent $\Rightarrow$ team loses closely (−5 €)
4. unmotivated players, strong opponent $\Rightarrow$ team loses (−10 €)

Whenever the team wins, it probably becomes motivated and, of course, whenever it loses it might become unmotivated. The exact transition matrix is as follows:

$$P = \begin{pmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

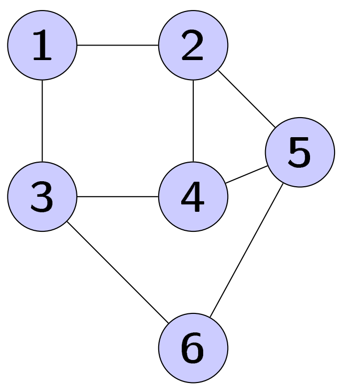
The initial distribution is uniform.

a) Write a Matlab program to simulate this Markov chain and estimate the average amount of money you win/lose per match using a sample size of $N = 10^5$.

b) Plot the running average of this amount of money, so if Y_i is what you win/lose in the i-th match, plot $Z_1, \ldots, Z_N$ with $Z_i = \frac{1}{i}\sum_{k=1}^{i} Y_k$. When would you say stationarity is reached (more or less)?

Exercise 4 (programming) (4 + 1 points)

Suppose you want to draw uniformly from all proper 5-colorings of the following graph.



a) Write a Matlab program to simulate this using a Gibb's sampler.

b) Estimate the fraction of proper 5-colorings of this graph that use only 4 colors or less with a sample size of at least $N = 10^5$.