Exercise 1-1 (4 points)

Prove the existence of a random field \( X = \{ X(t), t \in T \} \) with the following finite dimensional distributions and specify the measurable spaces \( (E_{t_1}, \ldots, t_n), (E_{t_1}, \ldots, t_n) \).

1. (2 points) \( X(t) \sim \text{Poi}(\lambda_t), \lambda_t > 0, t \in T \) and \( X(s), X(t) \) are independent for every \( t, s \in T, s \neq t \).

2. (2 points) \( X(t) = |Y(t)|^2 \), where \( Y(t) \sim N(0, I_n) \), \( I_n \) denotes the identity matrix and \( Y(s), Y(t) \) are independent for every \( t, s \in T, s \neq t \). Find the distribution of \( X(t) \).

Hint: Sometimes it’s easier to use Kolmogorov’s existence theorem formulated in terms of characteristic functions. For example, one can use Proposition A. \(^1\)

Exercise 1-2 (2 points)

Give an example (not the one presented in the lecture) of a non-continuous random function which has a continuous modification.

Exercise 1-3 (4 points)

Let \( Y \sim U[0, 1] \) and \( Z \) be a \( d \)-dimensional random vector, where \( Z \) and \( Y \) are independent. Consider a random field \( X = \{ X(t), t \in \mathbb{R}^d \} \) defined by \( X(t) = \sqrt{2} \cos(2\pi Y + \langle t, Z \rangle) \), where \( \langle \cdot, \cdot \rangle \) denotes the scalar product. Each realization of \( X \) is a cosine wave function.

1. (2 points) Compute the expectation of \( X(t) \) for \( t \in \mathbb{R}^d \).

2. (2 points) Determine the covariance function of \( X \).

Exercise 1-4 (2 points)

Let \( \Phi \) be a homogeneous Poisson point process in \( \mathbb{R}^d \) with intensity \( \lambda > 0 \).

1. (1 point) Write down the finite dimensional distributions of \( \Phi \) for disjoint bounded Borel sets \( B_1, \ldots, B_n \).

2. (1 point) Compute the expectation \( \mathbb{E}\Phi(B) \) and the variance \( \text{Var}\Phi(B) \) for bounded Borel sets \( B \).

\(^1\)Proposition A. The family of measures \( P_{t_1, \ldots, t_d} \) on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), (t_1, \ldots, t_d) \in T^d, d \geq 1 \), satisfies the conditions of symmetry and consistency iff for all \( d \geq 2, (s_1, \ldots, s_d) \in \mathbb{R}^d \) and \( (t_1, \ldots, t_d) \in T^d \) it holds

\[
\varphi_{P_{t_1, \ldots, t_d}}((s_1, \ldots, s_d)) = \varphi_{P_{t_1, \ldots, t_d}}((s_{i_1}, \ldots, s_{i_d}))
\]

for any permutation \( (1, \ldots, d) \rightarrow (i_1, \ldots, i_d) \),

\[
\varphi_{P_{t_1, \ldots, t_{d-1}}}((s_1, \ldots, s_{d-1})) = \varphi_{P_{t_1, \ldots, t_d}}((s_1, \ldots, s_{d-1}, 0)).
\]
Exercise 1-5 (5 points)

Let $\Phi$ be a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$. Consider the Shot-Noise-Field $\{X(t), t \in \mathbb{R}^d\}$ defined by

$$X(t) = \sum_{x \in \Phi} g(t - x),$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is a deterministic function with $\int_{\mathbb{R}^d} |g(z)| dz < \infty$ and $\int_{\mathbb{R}^d} g^2(z) dz < \infty$.

1. (2 points) Prove that $E[X(t)] = \lambda \int_{\mathbb{R}^d} g(z) dz$, $\forall t \in \mathbb{R}^d$.

2. (3 points) Prove that $\text{Cov}[X(s), X(t)] = \lambda \int_{\mathbb{R}^d} g(t - s - z)g(-z) dz$, $\forall s, t \in \mathbb{R}^d$.

Hint: Firstly, prove the statements for a simple function $g$. 