Exercise sheet 6 (total — 15 points) till June 30, 2016

Exercise 6-1 (3 points)
Let \( \tilde{f}, \tilde{g} \) be Laplace transforms of functions \( f, g : \mathbb{R}_+ \to \mathbb{R}_+ \).

1. (1 point) Find the Laplace transform of the convolution \( f * g \).

2. (2 points) Prove the final value theorem: \( \lim_{s \to 0} s \tilde{f}(s) = \lim_{t \to \infty} f(t) \).

Exercise 6-2 (2 points)
Let \( \{X_n\}_{n \geq 0} \) be i.i.d. r.v.’s with a density symmetric about 0 and continuous and positive at 0. Applying the Theorem 2.8 from the lecture notes, prove that cumulative distribution function \( F(x) := \mathbb{P}(X_1 \leq x), x \in \mathbb{R} \) belongs to the domain of attraction of a stable law \( G \). Find its parameters \( (\alpha, \lambda, \beta, \gamma) \) and sequences \( a_n, b_n \) s.t. \( \frac{1}{b_n} \sum_{i=1}^{n} X_i - a_n \xrightarrow{d} Y \sim G \) as \( n \to \infty \).

Exercise 6-3 (3 points)
Let \( \{X_n\}_{n \geq 0} \) be i.i.d. r.v.’s with for \( x > 1 \)

\[
\mathbb{P}(X_1 > x) = \theta x^{-\delta}, \quad \mathbb{P}(X_1 < -x) = (1 - \theta) x^{-\delta},
\]

where \( 0 < \delta < 2 \). Applying the Theorem 2.8 from the lecture notes, prove that c.d.f. \( F(x) := \mathbb{P}(X_1 \leq x), x \in \mathbb{R} \) belongs to the domain of attraction of a stable law \( G \). Find its parameters \( (\alpha, \lambda, \beta, \gamma) \) and sequences \( a_n, b_n \) s.t. \( \frac{1}{b_n} \sum_{i=1}^{n} X_i - a_n \xrightarrow{d} Y \sim G \) as \( n \to \infty \).

Exercise 6-4 (5 points)
Let \( X \) be a random variable with probability density function \( f(x) \). Assume that \( f(0) \neq 0 \) and that \( f(x) \) is continuous at \( x = 0 \). Prove that

1. (2 points) if \( 0 < r \leq \frac{1}{2} \), then \( |X|^{-r} \) belongs to the domain of attraction of a Gaussian law,

2. (3 points) if \( r > 1/2 \) then \( |X|^{-r} \) belongs to the domain of attraction of a stable law with stability index \( 1/r \).

Exercise 6-5 (2 points)
Find a distribution \( F \) which has infinite second moment and yet it is in the domain of attraction of the Gaussian law.