Exercise sheet 7 (total — 14 points) till July 14, 2016

Exercise 7-1 (3 points)
Prove the following statement which is used in the proof of Proposition 2.3 in the Lecture notes.
Let \( X \sim S_\alpha(\lambda, \beta, 0) \) with \( \alpha \in (0, 2) \). Then there exist two i.i.d. r.v.’s \( Y_1 \) and \( Y_2 \) with common distribution \( S_\alpha(\lambda, 1, 0) \) s.t.
\[
X \overset{d}{=} \begin{cases} 
\left( \frac{1+\beta}{2} \right)^{1/\alpha} Y_1 - \left( \frac{1-\beta}{2} \right)^{1/\alpha} Y_2, & \text{if } \alpha \neq 1, \\
\left( \frac{1+\beta}{2} \right) Y_1 - \left( \frac{1-\beta}{2} \right) Y_2 + \frac{\lambda}{\alpha} (1+\beta) \log \frac{1+\beta}{2} - (1-\beta) \log \frac{1-\beta}{2}, & \text{if } \alpha = 1.
\end{cases}
\]

Exercise 7-2 (3 points)
Prove that for \( \alpha \in (0, 1) \) and fixed \( \lambda \), the family of distributions \( S_\alpha(\lambda, \beta, 0) \) is stochastically ordered in \( \beta \), i.e., if \( X_\beta \sim S_\alpha(\lambda, \beta, 0) \) and \( \beta_1 \leq \beta_2 \) then \( P(X_{\beta_1} \geq x) \leq P(X_{\beta_2} \geq x) \) for \( x \in \mathbb{R} \).

Exercise 7-3 (3 points)
Prove Exercise 2.9 in the Lecture Notes: Show that if \( \frac{b_n}{n} \mu(b_n x) \sim C x^{-\alpha}, n \to \infty \), where \( \mu(x) = \int_{-\infty}^{\infty} y^2 dF(y) \), then
\[
\begin{align*}
n(F(b_n x) - 1) &\to c_1 x^{-\alpha}, \\
nF(-b_n x) &\to c_2 x^{-\alpha},
\end{align*}
\]
as \( n \to \infty \).

Exercise 7-4 (5 points)
Prove the following theorem.

**Theorem 1.** A distribution function \( F \) is in the domain of attraction of a stable law with exponent \( \alpha \in (0, 2) \) if and only if there are constants \( C_+, C_- \geq 0, C_+ + C_- > 0 \), such that

1. \[
\lim_{y \to +\infty} \frac{F(-y)}{1 - F(y)} = \begin{cases} 
C_- / C_+, & \text{if } C_+ > 0, \\
+\infty, & \text{if } C_+ = 0,
\end{cases}
\]

2. and for every \( a > 0 \)
\[
\begin{align*}
\lim_{y \to +\infty} \frac{1 - F(ay)}{1 - F(y)} &= a^{-\alpha}, & \text{if } C_+ > 0, \\
\lim_{y \to +\infty} \frac{F(-ay)}{F(-y)} &= a^{-\alpha}, & \text{if } C_- > 0.
\end{align*}
\]