Exercise sheet 1
till May 5, 2017

Please email your code to vitalii.makogin@uni-ulm.de

Theory (total – 13 points)

Exercise 1-1 (1+1+1 points)

Write down the transition matrices of the Markov chains, given by transition graphs:

a) 
\[
\begin{pmatrix}
0.6 & 0.4 \\
0.2 & 0.8 \\
0.4 & 0.6 & 0.2 \\
0.9 & 0.1 & 0.8 & 0.2 \\
\end{pmatrix}
\]

b) 
\[
\begin{pmatrix}
0.2 & 0.4 & 1 \\
0.6 & 0.4 & 0.1 \\
0.3 & 0.9 & 0.1 & 0.4 \\
0.1 & 0.2 & 0.3 & 0.4 & 0.1 \\
\end{pmatrix}
\]

c) 
\[
\begin{pmatrix}
p_0 & p_0 & p_0 & \ldots \\
1-p_0 & 1 & 1-p_0 & 1-p_0 & \ldots \\
1 & 0 & 0 & \ldots \\
\end{pmatrix}
\]

Exercise 1-2 (1+1+1+1 points)

Draw the transition graphs of of the Markov chains, given by transition matrices:

a) 
\[
\begin{pmatrix}
p_1 & p_2 & p_3 & \ldots \\
1 & 0 & 0 & \ldots \\
1 & 0 & 0 & \ldots \\
\end{pmatrix}
\]

b) 
\[
\begin{pmatrix}
p_1 & p_2 & p_3 & p_4 & \ldots \\
0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 1 & \ldots \\
\end{pmatrix}
\]

c) 
\[
\begin{pmatrix}
p_1 & p_2 & p_3 & \ldots \\
1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
\end{pmatrix}
\]

d) 
\[
\begin{pmatrix}
(1-p_1) & p_1 & 0 & 0 & \ldots \\
(1-p_2) & 0 & p_2 & 0 & \ldots \\
(1-p_3) & 0 & 0 & p_3 & \ldots \\
\end{pmatrix}
\]
Exercise 1-3 (1+2+2+1 points)

The transition matrix of a Markov chain \( \{X_n, n \geq 0\} \) is

\[
P = \begin{pmatrix}
0.1 & 0.2 & 0.7 \\
0.3 & 0.4 & 0.3 \\
0.5 & 0.4 & 0.1
\end{pmatrix}.
\]

The initial distribution is: \( P(X_0 = 0) = 0.6, P(X_0 = 1) = 0.3, P(X_0 = 2) = 0.1 \).

(1) Find the distribution of \( X_1 \).
(2) Find the probabilities \( P(X_2 = 0), P(X_1 = 0, X_2 = 1, X_3 = 2 | X_0 = 0), P(X_1 = 0, X_3 = 0, X_4 = 2 | X_0 = 1), P(X_1 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 = 0, X_8 = 0 | X_0 = 0) \).
(3) Find the probabilities \( P(X_0 = 0, X_1 = 1), P(X_1 = 0, X_2 = 1), P(X_4 = 0, X_3 = 1, X_1 = 0, X_2 = 2, X_5 = 0 | X_0 = 1) \).
(4) Find the distribution of the random variable \( \tau_1 = \inf\{n \geq 0 | X_n \neq 0\} \).

Programming (total – 8 points)

Exercise 1-4 (4 points)

Let r.v. \( Y \) has the cumulative distribution function \( F \) and r.v. \( X \sim U[0, 1] \). Prove that \( F^{-1}(X) \overset{d}{=} Y \). Using this fact, write the R program to simulate random variables with the following distributions.
1. Gaussian with \( \mu = 1, \sigma^2 = 1 \),
2. Poisson(1),
3. Exp(1).

Simulate the samples of size \( N = 1000 \). Compare them with samples generated by standard R functions \( \text{rnorm}, \text{rpois}, \text{rexp} \).

Exercise 1-5 (2+2 points)

A flea lives in a house with four dogs. Every day, it either stays where it is (with probability 0.7) or jumps (with probability 0.3) to one of the other dogs (selected uniformly).

(a) Write a R program to simulate this Markov chain using the initial distribution \( \mu_0 = \delta_1 \). Run your simulation for \( n = 365 \) days and plot a histogram of the visited states.

(b) Calculate the distribution of the flea’s position after 5, 10 and 365 days. Use the initial distribution from a).