Exercise sheet 2 till May 12, 2017

Please email your code to vitalii.makogin@uni-ulm.de till 8am, May 12

Theory (total – 11 points)

Exercise 2-1 (1+2 points)

Consider a Markov chain with state space $E = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
$$

Find a random mapping representation of $P$ using

(a) $Z \sim U(0, 1)$.

(b) $Z \sim \text{Bin}(3, \frac{1}{2})$, i.e., $Z$ binomial with parameters $n = 3$ and $p = \frac{1}{2}$.

Exercise 2-2 (2+1 points)

Consider a Markov chain whose transition matrix, $P$, is defined by the following graph

(a) Find the communicating classes of $P$. Which of these classes are closed?

(b) State if the transition matrix is irreducible and justify your answer.

Exercise 2-3 (3 points)

Suppose that a Markov chain $\{X_n, n \geq 0\}$ visits the set $A \subset X$ infinitely many times with probability 1. Let $\tau_m$ be the $m$th moment. That is, $\tau_m = \inf\{k > \tau_{m-1} | X_k \in A\}$, where $\tau_{-1} \equiv 0$. Prove that $Y_n = X_{\tau_n}, n \geq 0$ is a Markov chain.
Exercise 2-4 (3 points)

Let \( \{\varepsilon_n, n \in \mathbb{N}\} \) be a sequence of independent Bernoulli random variables, i.e.,
\[
P(\varepsilon_n = 1) = p, \quad P(\varepsilon_n = -1) = 1 - p.
\]
For which \( p \) is a sequence \( \{X_n := \varepsilon_{n+1} \varepsilon_n, n \in \mathbb{N}\} \) a Markov chain?

Programming (total – 8 points)

Exercise 2-5 (4 points)

Consider the Markov chain \( \{Y_n\}_{n \in \mathbb{N}} \) with initial distribution \( \mu_0 = \delta_1 \) and the following transition graph.

![Transition Graph](image)

Consider the following random variables:

(i) \( \tau_1 = \inf\{n \geq 0 : Y_n + Y_{n+1} = 5\} \)

(ii) \( \tau_2 = \inf\{n \geq 0 : Y_{\lceil n/2 \rceil} \geq 4\} \)

(iii) \( \tau_3 = \sup\{n \geq 0 : Y_n \in \{1, 2, 3\}\} \)

Write a R program to simulate \( \{Y_n\}_{n \in \mathbb{N}} \) and estimate the expectation of \( \tau_1, \tau_2 \) and \( \tau_3 \) if they are stopping times of \( \{Y_n\}_{n \in \mathbb{N}} \). Use a sample size of at least \( N = 10^4 \).

Exercise 2-6 (4 points)

Consider the Markov chain \( \{X_n\}_{n \in \mathbb{N}} \) with state space \( \mathbb{Z} \), initial distribution \( \mu_0 = \delta_0 \) and transition probabilities
\[
p_{i,j} = \begin{cases} 
1, & \text{if } j = i - 1, \\
\frac{1}{2}, & \text{if } j = i, \\
\frac{1}{4}, & \text{if } j = i + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Simulate \( \{X_n\}_{n \in \mathbb{N}} \) and draw one realization of the first 10 steps.