Exercise sheet 6 till June 9, 2017

Theory (total – 11 points)

Exercise 6-1 (3 points)

Prove that the Metropolis-Hastings algorithm works, i.e., prove that (for any proposal matrix $Q$) the transition matrix of the Markov chain used in the Metropolis-Hastings algorithm is in detailed balance with $\pi$.

Exercise 6-2 (3 points)

Consider a Markov chain with state space $\mathcal{X}$ and transition matrix $P$, and two arbitrary probability measures, $\mu$ and $\nu$, on $\mathcal{X}$. Show that

$$\|\mu P - \nu P\|_{TV} \leq \|\mu - \nu\|_{TV}$$

and explain why this means that a Markov chain can only get closer to its stationary distribution, assuming that one exists.

Exercise 6-3 (5 points)

In Barker’s algorithm for Markov Chains one takes the acceptance probability as

$$\alpha_{i,j} = \frac{\lambda_j}{\lambda_i + \lambda_j}.$$

Show that a Metropolis-Hastings algorithm with this acceptance probability generates a Markov Chain with the target distribution as stationary distribution under a certain condition (which ?) on matrix $Q$.

Programming (total – 8 points)

Exercise 6-4 (4 points)

Consider the following graph.

The graph is coloured by red and green such that no two neighboring vertices are red. Write the R code to generate the uniform sample ($n = 100$) from all possible appropriate colouring of this graph.
Exercise 6-5 (4 points)

Construct by the Metropolis algorithm the transition matrix $P$ of a Markov chain with state space $E = \{0, 1, \ldots, l\}$ and limit distribution

$$
\pi_i = \frac{\mu^i \exp(-\mu)}{z \cdot i!}, \quad \forall i \in E,
$$

where $\mu \in (0, 1)$ and $z$ is a normalizing constant of the probability mass function. A potential successor of state $i$ should be chosen uniformly on $\{i-1, i+1\}$ (set $-1 = 0, l + 1 = l$).