Exercise 1 (theory) (2 points)

Suppose we have a “black box” which on command can generate the value of a gamma random variable with parameters $\alpha = \frac{3}{2}$ and $\lambda = 1$. Explain how we can use this black box to approximate $E((X + 1)^{-2})$, where $X$ is an exponential random variable with mean 1. Write out your estimator and show that it is unbiased.

Recall that the Gamma distribution with parameters $\alpha, \lambda > 0$ has the density

$$f(x; \alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} I\{x > 0\}.$$ 

Exercise 2 (theory) (1 + 1 + 2 points)

Consider a random variable $X \sim \text{Exp}(\lambda)$ with $\lambda \in (0, \infty)$. Estimate $P(X > 10)$ by means of importance sampling. Use exponential tilting, i.e., assume that the importance sampling density $g(x)$ belongs to the following parametric family of distributions:

$$\{g(\cdot; \theta) : \theta \in (-\infty, \lambda)\} \quad \text{with} \quad g(x; \theta) = e^{\theta x - \kappa(\theta)} f(x) \quad \text{and} \quad \kappa(\theta) = \log \left( E e^{\theta X} \right).$$

a) Calculate the probability $P(X > 10)$ by hand and write out the importance sampling density $g(x; \theta)$.

b) Find the zero variance density $g^*(x)$.

c) Find the variance minimization parameter $\theta_{\text{VM}}$ for this problem (by hand).

Exercise 3 (programming) (1 + 2 + 3)

Consider the setting described in Exercise 2. Write a Matlab program to estimate $P(X > 10)$ for $\lambda = \frac{1}{2}$ using

a) standard Monte Carlo

b) importance sampling with $\theta_{\text{VM}}$ as calculated in Exercise 2

c) importance sampling, estimating $\theta_{\text{VM}}$ using an initial sample
and compare the estimators by estimating their relative errors. Use a sample size of at least $N = 10^5$ in each case.

**Exercise 4 (theory)**  (2 + 3 points)

Let $N$ be a random variable with values in $\mathbb{N}$ and let $Y_1, Y_2, \ldots$ be i.i.d. random variables which are independent of $N$. Consider

$$X = \sum_{i=1}^{N} Y_i,$$

i.e., the sum of a random number of random variables.

a) Show that $\mathbb{E}X = \mathbb{E}N \cdot \mathbb{E}Y_1$.

b) Let $g_N$ be the probability generating function of $N$ and let $M_Y$ be the moment generating function of $Y_1$. Show that the moment generating function of $X$ is

$$M_X(t) = g_N(M_Y(t)).$$

**Exercise 5 (theory)**  (4 points)

Consider a discrete-time stochastic process $\{X_n\}_{n \in \mathbb{N}}$ with values in $\mathbb{Z}$. Which of the following random variables are stopping times?

(i) $\tau_1 = \inf \{n \geq 0 : X_n + X_0 = 5\}$

(ii) $\tau_2 = 10$

(iii) $\tau_3 = \sup \{n \geq 0 : X_n \in \{1, 2, 3\}\}$

(iv) $\tau_4 = \inf \{n \geq 0 : Y_{\lceil n/2 \rceil} \geq 4\}$

Justify your answer.

**Exercise 6 (programming)**  (2 + 3 points)

Consider an incompetent businessman. His company starts off with 10000 € but makes a loss, on average, each day. More precisely, the profit or loss on the $i$th day is described by a random variable $Y_i \sim N(-30, 10000)$. If his company can get 11000 € in the bank, he is able to sell his company to a competitor. If his company’s bank account drops below 0 €, he goes bankrupt.

a) Write a Matlab program to estimate the probability that the business man sells his company before he goes bankrupt using standard Monte Carlo and a sample size of at least $N = 10^4$. Estimate the standard deviation of your estimator.

b) The event that the business man sells his company before he goes bankrupt happens quite rarely. Write a Matlab program to estimate this probability using importance sampling. Estimate the standard deviation of your estimator and compare it to your results from a).

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