Stochastics II
Exercise Sheet 15
Due to: Wednesday, 10th of February 2016

Exercise 1 (5 Points)
Let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of random variables, such that \( S_n := \sum_{k=1}^{n} X_k \) converges almost surely for \( n \to \infty \). Furthermore let \( \{a_n\}_{n \in \mathbb{N}} \) be a monotonously increasing sequence of non-negative real numbers with \( a_n \xrightarrow{n \to \infty} \infty \). Show\(^1\) that
\[
\frac{1}{a_n} \sum_{k=1}^{n} a_k X_k \to 0, \quad \text{a.s.}
\]
as \( n \to \infty \).

Exercise 2 (3 Points)
Let \( X \) be a non-negative random variable on some probability space \((\Omega, \mathcal{F}, P)\) and \( T : \Omega \to \Omega \) a measure preserving map. Show\(^2\) that
\[
\sum_{k=1}^{\infty} X(T^k(\omega)) = \infty,
\]
for almost all \( \omega \in \Omega \) with \( X(\omega) > 0 \).

Exercise 3 (4 Points)
Let \( X \) be a non-negative random variable on some probability space \((\Omega, \mathcal{F}, P)\) and \( T : \Omega \to \Omega \) a measure preserving map. Show\(^3\) that \( E(X) = E(X \circ T) \), i.e.
\[
\int_{\Omega} X(T(\omega)) P(d\omega) = \int_{\Omega} X(\omega) P(d\omega).
\]

Exercise 4 (4 Points)
Let \((\Omega, \mathcal{F}, P) = ([0,1), \mathcal{B}([0,1]), \nu)\), where \( \nu \) denotes the Lebesgue measure on \([0,1)\). Let \( \lambda \in (0,1) \).

(a) Show that \( T(x) = (x + \lambda) \) (mod 1) is a measure preserving map, where \( a \) (mod \( b \)) := \( a - \left\lfloor \frac{a}{b} \right\rfloor \cdot b \) for \( a \in \mathbb{R} \) and \( b \in \mathbb{Z} \).

(b) Show that \( T(x) = \lambda x \) and \( T(x) = x^2 \) are not measure preserving.

Note: We will provide one more exercise till Tuesday.

\(^1\)It holds \( \frac{1}{a_n} \sum_{k=1}^{n} a_k X_k = \frac{1}{a_n} \sum_{k=1}^{n} (a_k - a_{k-1})(S_n - S_{k-1}) \), \( a_0 := 0 \), \( S_0 := 0 \).
\(^2\)Use Poincaré’s Theorem (Th. 6.2.1).
\(^3\)Use algebraic induction.