Stochastics II
Exercise Sheet 4
Due to: Wednesday, 11th of November 2015

Exercise 1 (2 Points)
Prove Remark 2.1.4: For a delayed renewal-process $N = \{N(t); \ t \geq 0\}$ with delay $T_1$ it holds

(a) $H(t) = \sum_{k=0}^{\infty} \left( F_{T_1} \ast F_{T_2}^k \right)(t), \ t \geq 0$. 
(b) $\hat{H}(s) = \frac{\hat{T}_1(s)}{1-\hat{T}_2(s)}, \ s > 0$.

Exercise 2 (2 Points)
Let $N = \{N(t); \ t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Find

(a) $P(N_1 = 2, N_2 = 3, N_3 = 5)$. 
(b) $P(N_1 \leq 2, N_2 = 3, N_3 \geq 5)$. 
(c) the probability that $N(t)$ is even respectively odd, $t > 0$.

Exercise 3 (4 Points)
Let $N = \{N(t); \ t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Calculate

$$P(N(s) = i|N(t) = n)$$

for $0 < s < t$ and $i = 1, \ldots, n$.

Exercise 4 (6 Points)
Let $N = \{N(t); \ t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Let $Y$ be a random variable with $P(Y = 1) = P(Y = -1) = 1/2$ which is independent of the process $N$. Define a stochastic process $X = \{X(t); \ t \geq 0\}$ by $X(t) = Y \cdot (-1)^N(t)$.

(a) Let $t > 0$ be arbitrary but fixed. Calculate the probability that $X(t)$ is $1$ ($-1$, respectively).
(b) Calculate the covariance function of $X$.

Exercise 5 (8 Points)
Let $\mathcal{B}_0(\mathbb{R}^d) := \{B \in \mathcal{B}(\mathbb{R}^d), \ \nu_d(B) < \infty\}$, where $\nu_d$ denotes the $d$-dimensional Lebesgue-measure. Let furthermore $\mu : \mathcal{B}(\mathbb{R}^d) \rightarrow [0, \infty]$ be a locally finite measure (i.e. $\mu(B) < \infty$ for every $B \in \mathcal{B}_0(\mathbb{R}^d)$). The non-homogeneous Poisson process can be defined as $N = \{N(B), \ B \in \mathcal{B}(\mathbb{R}^d)\}$ with the following properties:
1.) \( N(B) \sim Poi(\mu(B)) \) for every \( B \in \mathcal{B}_0(\mathbb{R}^d) \).

2.) \( N(B_1), \ldots, N(B_n) \) are independent random variables for pairwise disjoint \( B_i \in \mathcal{B}_0(\mathbb{R}^d) \), \( i = 1, \ldots, n \) and arbitrary \( n \in \mathbb{N} \).

\( \mu \) is called the intensity measure of \( N \).

(a) Let \( A_1, A_2 \in \mathcal{B}_0(\mathbb{R}^d) \) be arbitrary. Show that for fixed \( k \in \mathbb{N}_0 \) and \( l = 0, \ldots, k \), the probability \( P(N(A_1 \cup A_2) = k, N(A_1 \cap A_2) = l) \) is given by

\[
\frac{\mu^k(A_1 \cup A_2)}{k!} e^{-\mu(A_1 \cup A_2)} \binom{k}{l} \left( \frac{\mu(A_1 \cap A_2)}{\mu(A_1 \cup A_2)} \right)^l \left( 1 - \frac{\mu(A_1 \cap A_2)}{\mu(A_1 \cup A_2)} \right)^{k-l}.
\]

(b) Let \( n \in \mathbb{N}, k_1, \ldots, k_n \in \mathbb{N}_0 \) and \( B_1, \ldots, B_n \in \mathcal{B}_0(\mathbb{R}^d) \) pairwise disjoint. Verify that for \( k = \sum_{i=1}^n k_i \) and \( B = \bigcup_{i=1}^n B_i \) it holds

\[
P(N(B_1) = k_1, \ldots, N(B_n) = k_n | N(B) = k) = \frac{k!}{k_1! \ldots k_n!} \frac{\mu^{k_1}(B_1) \ldots \mu^{k_n}(B_n)}{\mu^k(B)}
\]

provided that \( \mu(B) > 0 \).