Exercise 1 (6 Points)
Let \( X = \{X(t), \ t \geq 0\}, X(t) = \sum_{j=1}^{N(t)} U_j, \ t \geq 0, \) be a compound Poisson process with parameters \( \lambda, F_U. \)

(a) Determine its characteristic function \( \phi_{X(t)}(z) = \mathbb{E}e^{izX(t)}, \ z \in \mathbb{R}, \ t \geq 0. \)

(b) Let \( \mathbb{E}|U_1| < \infty. \) Show that
\[
\phi'_U(z) = \frac{1}{\lambda t} \phi_{X(t)}(z), \quad z \in \mathbb{R},
\]
where \( \phi_U \) denotes the characteristic function of \( U_1. \)

Exercise 2 (9 Points)
Let \( N = \{N(t), \ t \in [0, \infty]\} \) be a Cox process with random intensity measure \( \Lambda = \{\Lambda(B), \ B \in \mathcal{B}(\mathbb{R}^+)\}. \) Show the following statements:

(a) If the moment generating function \( M_{\Lambda((0,t])} \) exists in a neighborhood \( U \) of 0, then
\[
M_{N(t)}(s) = M_{\Lambda((0,t])}(e^s - 1), \quad s \in \tilde{U},
\]
where \( M_{N(t)}(s) \) denotes the moment generating function of \( N(t), \ t > 0 \) and \( \tilde{U} = \{s \in \mathbb{R}, \ e^s - 1 \in U\}. \)

(b) If \( \mathbb{E}|\Lambda((0,t])| < \infty, \) then \( \mathbb{E}N(t) = \mathbb{E}\Lambda((0,t]). \)

(c) If \( \mathbb{E}\Lambda^2((0,t]) < \infty, \) then \( \text{Var}N(t) = \mathbb{E}\Lambda((0,t]) + \text{Var}\Lambda((0,t]). \)

Exercise 3 (3 Points)
Give an algorithm of how to simulate the trajectories of a Wiener process \( W = \{W(t), \ t \in [0,1]\} \) by using the independence and the distribution of the increments of \( W. \)

Exercise 4 (4 Points)
An approximation for the Wiener process \( W = \{W(t), \ t \in [0,1]\} \) is given by
\[
W_n(t) = \sum_{k=1}^{n} S_k(t)Z_k
\] (1)
where $S_k$ are the Schauder functions, $t \in [0, 1]$, $k \geq 1$ and $Z_k \overset{i.i.d.}{\sim} N(0, 1)$. The convergence of the sequence in (1) ($n \to \infty$) has to be understood in $L^2$-sense for all $t \in [0, 1]$. Show\(^1\) that

$$W_n(t) \overset{L^2}{\to} W(t) \quad (n \to \infty)$$

\(^1\)You can use that the $L^2$-limit of $\{W_n(t)\}_{n \in \mathbb{N}}$ is a Wiener process without proof (see also Theorem 3.2.1.).