1. Exercise sheet
Deadline: November, 4th, 12:15

Exercise 1: Is the origin in the convex hull of random points?
(4 Credits)
Let \( K \subseteq \mathbb{R}^2 \) be a convex and compact set with \( 0 \in \text{int} \, K \). Let \( X_1, \ldots, X_n \) be independent random points distributed uniformly in \( K \). Show that there is a constant \( c < 1 \) (depending on \( K \)) with
\[
P(0 \notin \text{conv}\{X_1, \ldots, X_n\}) \in O(c^n), \quad n \to \infty.
\]

Exercise 2: The line \( g(\varphi, p) \)
(2+4+6 Credits)
Here we want to examine the line \( g(\varphi, p) \) introduced in the lecture (in order to prepare an example we will treat on November, 2nd). Recall that \( g(\varphi, p) \) is for \( \varphi \in [0, 2\pi) \) and \( p > 0 \) the line whose normal vector pointing away from 0 forms an angle of \( \varphi \) with the first unit vector (measured counter-clockwise from the first unit vector to the normal vector) and which has distance \( p \) from the origin.

a) If \( g(\varphi, p) \) is not parallel to the second unit vector, it is the graph of an affine function. Determine this function.

Now let \( K = [-\frac{a}{2}, \frac{a}{2}]^2 \) for \( a > 0 \) be the axis-parallel square centered at the origin of side-length \( a > 0 \). Assume from now on \( 0 < \varphi < \pi/4 \).

b) Show that \( g(\varphi, p) \) intersects \( K \) if and only if \( p \leq \frac{a}{2}(\sin \varphi + \cos \varphi) \). Show that the intersection points of \( g(\varphi, p) \) with the boundary of \( K \) are
\[
\begin{cases}
\left(\frac{p}{\cos \varphi} - \frac{a}{2} \tan \varphi, \frac{a}{2}\right) \text{ and } \left(\frac{p}{\cos \varphi} + \frac{a}{2} \tan \varphi, -\frac{a}{2}\right) & \text{if } p \leq \frac{a}{2}(\cos \varphi - \sin \varphi) \\
\left(\frac{p}{\cos \varphi} - \frac{a}{2} \tan \varphi, \frac{a}{2}\right) \text{ and } \left(\frac{p}{\sin \varphi} - \frac{a}{2} \tan \varphi, \frac{a}{2}\right) & \text{if } p \in \left[\frac{a}{2}(\cos \varphi - \sin \varphi), \frac{a}{2}(\cos \varphi + \sin \varphi)\right].
\end{cases}
\]

Exercise 3: The number of edges using \( R \)
(2+4+1+1+3+1=12 Credits)
Again, let \( X_1, \ldots, X_n \) be independent random points distributed uniformly in some convex and compact set \( K \subseteq \mathbb{R}^2 \) with interior points. We want to examine the number of vertices - or equivalent the number of faces - of \( \text{conv}\{X_1, \ldots, X_n\} \).

a) What can you tell (from the results of the lecture) about the asymptotic behavior of the expected value of the number of faces when \( K \) is a square and when \( K \) is a ball.

b) Write a function that determines for points \( x_1, \ldots, x_n \subseteq \mathbb{R}^2 \) the number of faces of \( \text{conv}\{x_1, \ldots, x_n\} \). Hint: Implement the indicator \( \epsilon_{ij} \) from the proof of the lemma in the lecture. For this, use that a segment from a point \( x_i \) to a point \( x_j \) is an edge of \( \text{conv}\{x_1, \ldots, x_n\} \) if and only if \( i \) and \( j \) are minimizers or maximizers of the map \( k \mapsto (x_k, n_{ij}) \), where \( n_{ij} \) is a vector perpendicular to the line segment from \( x_i \) to \( x_j \).

c) Apply the function from part b) to 100 points \( X_1, \ldots, X_{100} \) chosen uniformly from the unit square \( K = [0, 1]^2 \).

d) Repeat part c) 1000 times. Use the results to estimate mean and variance of the number of faces and plot a histogram.

e) Repeat part c) and d) for the unit ball
\[
K := B_1(0) := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.
\]

Hint: In order to simulate a random point distributed uniformly in the unit ball proceed as follows: Simulate a point \( X \) distributed uniformly in the square \([-1, 1]^2\). If \( X \in B_1(0) \), you are done. Otherwise repeat this until you get a point in \( B_1(0) \).

f) Compare the results from part d) and e) in the view of a).