Exercise sheet 13 (total – 24 points) till February 8, 2017

Exercise 13-1 (3 points)

Let $F_n, n \geq 0$ be a filtration on $(\Omega, \mathcal{F}, P)$. Let $\sigma$ and $\tau$ be two stopping times w.r.t. $F_n, n \geq 0$ such that $\sigma(\omega) \leq \tau(\omega), \omega \in \Omega$ and $A_n = \{ \omega : \sigma(\omega) < n \leq \tau(\omega) \}$. Show that $A_n$ is $F_{n-1}$-measurable for each $n \geq 1$.

Exercise 13-2 (3 points)

Let a random sequence $\{X_n, n \geq 0\}$ be a submartingale w.r.t. the filtration $\{F_n, n \geq 0\}$. Prove that there exists a non-negative martingale $\{M_n, n \geq 0\}$ such that

$$\max(0, X_n) \leq M_n, n \geq 0, \quad \text{and} \quad \sup_{n \geq 0} E[\max(0, X_n)] = \sup_{n \geq 0} EM_n.$$

Exercise 13-3 (4 points)

*(Doob decomposition)* Let a random sequence $\{X_n, n \geq 0\}$ be a submartingale w.r.t. the filtration $\{F_n, n \geq 0\}$. Prove that there exists a martingale $\{M_n, n \geq 0\}$ and non-decreasing integrable random sequence $\{A_n, n \geq 0\}$ such that $A_0 = 0$, $A_n$ is $F_{n-1}$-measurable for each $n \geq 1$, and

$$X_n = M_n + A_n, \forall n \geq 0.$$

Exercise 13-4 (3 points)

*(Polya’s Urn Scheme)* An urn contains $r$ red and $g$ green balls. At each time, we draw a ball out, then put it back, and add $c$ more balls of the color drawn. Let $X_n$ be the fraction of green balls after the $n^{th}$ drawing. Prove that $X_n, n \geq 1$ is a martingale.

Exercise 13-5 (5 points)

*(L$_p$ maximum inequality)* If $X_n, n \geq 0$ is a martingale then for any $1 < p < \infty$

$$E \left( \max_{0 \leq n \leq N} |X_n| \right)^p \leq \left( \frac{p}{p-1} \right)^p E(|X_N|^p).$$

Hint: Firstly, consider $(\max_{0 \leq n \leq N} X_n) \wedge M$ for some constant $M > 0$. Then apply Doob’s inequality for it.

Exercise 13-6 (6 points)

(a) Prove that if $X_n, n \geq 0$ is a non-negative supermartingale, then $\exists \lim_{n \to \infty} X_n = X$ a.s. and $EX \leq EX_0$.

Hint: Use upcrossing inequality: if $Y_n, n \geq 0$ is a submartingale then $(b - a)E U_n \leq E(Y_n - a)_+ - E(Y_0 - a)_+$, where $U_n$ is the number of upcrossings of interval $(a, b)$ by $Y_m, m \geq 0$ completed by time $n$.

(b) Let random sequences $X_n, n \geq 0$ and $Y_n, n \geq 0$ be a.s. non-negative, integrable and adapted to the filtration $F_n, n \geq 0$. Suppose $E(X_{n+1} | F_n) \leq (1 + Y_n) X_n, \forall n \geq 0$ with $\sum_{n=0}^{\infty} Y_n < \infty$ a.s. Prove that $X_n$ converges a.s. to a finite limit.

Hint: Consider $Z_n = X_n / \prod_{i=1}^{n-1} (1 + Y_i), n \geq 1$. 
