Stochastics II
Universität Ulm
WS 2016/2017
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Exercise sheet 14 (total – 16 points) till February 14, 2017

Exercise 14-1 (4 points)
Let \((\Omega, \mathcal{F}, P) = ([0,1), \mathcal{B}([0,1]), \nu)\), where \(\nu\) denotes the Lebesgue measure on \([0,1)\). Let \(\lambda \in (0,1)\).

1. Show that \(T(x) = (x + \lambda) \pmod{1}\) is a measure preserving map, where \(a \pmod{b} := a - \lfloor \frac{a}{b} \rfloor \cdot b\) for \(a \in \mathbb{R}\) and \(b \in \mathbb{Z}\).

2. Show that \(T(x) = \lambda x\) and \(T(x) = x^2\) are not measure preserving.

Exercise 14-2 (2 points)
Let a stationary sequence \(X_n, n \geq 0\) be generated by a random variable \(X_0\) and a measure preserving map \(T\). Assume that \(X\) is \(m\)-dependent, that is, families of random variables \(\{X_k, k \leq n\}\) and \(\{X_j, j \geq n + m\}\) are independent for any \(n\). Prove that \(T\) is ergodic.

Exercise 14-3 (2 points)
Let \(X\) be a non-negative random variable on some probability space \((\Omega, \mathcal{F}, P)\) and \(T : \Omega \to \Omega\) a measure preserving map. Show\(^1\) that \(\mathbb{E}(X) = \mathbb{E}(X \circ T)\), i.e.

\[
\int_{\Omega} X(T(\omega)) P(d\omega) = \int_{\Omega} X(\omega) P(d\omega).
\]

Exercise 14-4 (6 points)
Let \(\Omega = \mathbb{R}^2\) and \(P\) be a normal distribution in \(\mathbb{R}^2\) with zero mean and identity matrix of covariances. Assume that transformation \(T : \Omega \to \Omega\) acts in polar coordinates as \(T((r, \varphi)) = (r, 2\varphi \pmod{2\pi}), r \geq 0, 0 \leq \varphi < 2\pi\).

1. (2 point) Prove that \(T\) preserves the measure \(P\).

2. (4 points) Find the limit

\[
\lim_{n \to \infty} \frac{1}{n} \left( \sum_{k=0}^{n-1} f(T^k(x)) \right), \quad x \in \mathbb{R}^2
\]

for \(f_1 = x_1^2, f_2(x) = x_1, x_2\).

Hint: At first, prove this fact for the functions of the form

\[
f(r, \varphi) = \sum_{k=0}^{m} c_k \mathbb{1}\{\varphi \in [\alpha_k, \beta_k]\} \mathbb{1}\{r \in [x_k, y_k]\},
\]

and then pass to a limit.

Exercise 14-5 (2 points)
Let \(X_n, n \geq 0\) be a centered Gaussian stationary sequence with covariance function \(C(n) = \mathbb{E}(X_n X_{n+k})\). Let \(C(n) \to 0, n \to \infty\). Prove that the measure preserving map \(T\), which corresponds to \(X\), i.e. \(X_n \overset{d}{=} X_0(T^n)\), is mixing (on average) and, consequently, ergodic.

\(^1\)Use algebraic induction.