Exercise sheet 3 (total – 17 points) till November 9, 2016

Let \{X(t), t \in \mathbb{R}_+\} be a real-valued stochastic process. Denote by \( \mu_X(t) = \mathbb{E}X(t), t \in \mathbb{R}_+ \), \( K_X(t, s) = \text{Cov}(t, s), t, s \in \mathbb{R}_+ \), the mean and covariance functions of the process \( X \), respectively.

Exercise 3-1 (3 points)
Find \( \mu_x \) and \( K_X \) of the process \( X(t) = \xi_1 f_1(t) + \cdots + \xi_n f_n(t), t \in \mathbb{R}, \) where \( f_1, \ldots, f_n \) are nonrandom functions, and \( \xi_1, \ldots, \xi_n \) are non-correlated random variables with means \( \mu_1, \ldots, \mu_n \) and variances \( \sigma_1, \ldots, \sigma_n \).

Exercise 3-2 (3 points)
Let \{\(X(t), t \in \mathbb{R}_+\)\} be a stochastic process with independent increments and \( \mathbb{E}|X(t)|^2 < \infty, t \in \mathbb{R}_+ \). Prove that its covariance function is equal to \( K_X(t, s) = F(t \wedge s), t, s \in \mathbb{R}_+, \) where \( F \) is some non-decreasing function.

Exercise 3-3 (3 points)
Let \( X, Y \) be two independent square integrable centered stochastic processes and \( c > 0 \) be a constant. Prove that \( K_{X+Y} = K_X + K_Y, K_{\sqrt{c}X} = cK_X, K_{XY} = K_XK_Y \).

Exercise 3-4 (4 points)
Let \{\(X(t), t \in \mathbb{R}_+\)\} be a real-valued process with stationary and independent increments, and \( \text{Var}(X(1) - X(0)) > 0 \). Prove that the processes \( X \) and \( Y(t) := X(t + 1) - X(t) \) are mean square continuous but not mean square differentiable.

Hint: You may use the solution of Cauchy’s functional equation \( f(x + y) = f(x) + f(y) \). If \( f \) is bounded on any interval then solution has a form \( f(x) = cx \), where \( c \in \mathbb{R} \).

Exercise 3-5 (4 points)
Let the mean and covariance functions of the process \( X \) be equal to \( \mu_X(t) = t^2, K_X(t, s) = e^{ts} \). Prove that \( X \) is mean square differentiable. Let \( X' \) be its derivative defined in \( L_2 \) sense. Please find:

1. \( \mathbb{E}|X(t) + X(s)|^2, t, s \in \mathbb{R} \).
2. \( \mathbb{E}[X(t)X'(s)], t, s \in \mathbb{R} \).