Exercise sheet 8 (total – 19 points) till December 14, 2016

Exercise 5-1 (2 points)
Let $W$ be the Wiener process. Find the characteristic function for $W(2) + 2W(1)$.

Exercise 5-2 (4 points)
Let $W$ be the Wiener process. Find:
1. $E(W(t))^m, m \in \mathbb{N}$,
2. $E\exp(2W(1) + W(2))$,
3. $E\cos(2W(1) + W(2))$,
4. $E[W(2) > 1|W(1) < 2]$.

Exercise 5-3 (3 points)
Let $X = \{X(t) := \int_0^t W(s) ds, t \geq 0\}$, where $W$ is the Wiener process. Find the distribution of random variable $X(T)$ for $T > 0$.

Hint: Recall that a limit of Gaussian random variables is also Gaussian.

Exercise 5-4 (10 points)
(6 points) Write a program in R which simulates the trajectory of the Wiener process on $[0, T]$ (a) by using approximation with Schauder functions and input parameters $t, T$ and $m$, where $t$ is a finite dimensional vector of locations in $[0, T]$ and $m$ is the cut-off parameter of the series expansion;
(a) by using the independence and the distribution of the increments of $W$, with input parameter $t$ defined as in (a);
(c) by using ideas for the proof of Theorem 3.3.1: Let $\{W(t), t \in [0, 1]\}$ be the Wiener process and $Z_1, Z_2, \ldots$ a sequence of independent random variables with $P(Z_i = 1) = P(Z_i = -1) = \frac{1}{2}$ for all $i \in \mathbb{Z}_+$. For every $n \in \mathbb{N}$ we define $\{\tilde{W}^n(t), t \in [0, 1]\}$ by $\tilde{W}^n(t) = \frac{S_i}{\sqrt{n}} + (nt - \lfloor nt \rfloor)\frac{Z_i}{\sqrt{n}}$, where $S_i = Z_1 + \ldots + Z_i$, $i \geq 1$, $S_0 = 0$.

(1 point) Simulate 500 trajectories of a Wiener process on $[0, 5]$ in cases (a)-(c). Take $m = 10$ in (a) and $t = (t_0, ..., t_{1000})$ in (b), where $t_0 = 0$ and $t_k = kT/1000, k = 1, ..., 1000$. Take $n = 1000$ in (c).
(1 point) Plot one trajectory for all cases.
(3 points) For each simulated trajectory $\tilde{W}$ compute the approximation $\tilde{X}(5) = \frac{1}{1000} \sum_{i=0}^{1000} \tilde{W}(t_i)$ of random variable $X(5)$. Compare the empirical distribution of $\tilde{X}(5)$ with the distribution of $X(5)$ from Exercise 5-3 using Kolmogorov-Smirnov test and Kolmogorov’s distance as a measure. Which method of simulation is better?