Stochastics II | **Universität Ulm**
WS 2017/2018 | Prof. Dr. Evgeny Spodarev
January 9, 2018 | Dr. Vitalii Makogin

Exercise sheet 10 (total – 19 points) till January 17, 2018

Exercise 10-1 (4 points)

Let $P(s) = \sum_{k=0}^{\infty} p_k s^k$ where $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Assume $P(0) = p_0 > 0$ and that $\log\left(\frac{P(s)}{P(0)}\right)$ is a power series with positive coefficients. If φ is a the characteristic function of an arbitrary distribution F show that $P(\varphi)$ is an ID characteristric function.

Hint: Find the Levy measure in terms of F^{*n}.

Exercise 10-2 (4 points)

Let φ be a characteristic function. Show that $\psi : \mathbb{R} \to \mathbb{C}$ defined by

$$\psi(z) = \frac{1-b}{1-a}\frac{1-a\varphi(z)}{1-b\varphi(z)}, 0 \leq a < b < 1,$$

is an ID characteristic function.

Exercise 10-3 (6 points)

Let $\{X(t), t \geq 0\}$ be a Lévy process with charactersistic Lévy exponent η and $\{\tau(s), s \geq 0\}$ an independent subordinator with characteristic Lévy exponent γ. The stochastic process Y be defined as $Y = \{X(\tau(s)), s \geq 0\}$.

1. (3 points) Show that

$$\mathbf{E}\left(e^{i\theta Y(s)}\right) = e^{\gamma(-i\eta(\theta))s}, \quad \theta \in \mathbb{R}.$$

   Hint: Since τ is a process with non-negative values, it holds $\mathbf{E}e^{i\theta\tau(s)} = e^{\gamma(\theta)s}$ for all $\theta \in \{z \in \mathbb{C} : \mathrm{Im} z \geq 0\}$ through the analytical continuation of Theorem 4.2.1.

2. (3 points) Show that Y is a Lévy process with characteristic Lévy exponent $\gamma(-i\eta(\cdot))$.

Exercise 10-4 (2 points)

Let W be a standard Wiener process and τ an independent $\frac{\alpha}{2}-$stable subordinator, where $\alpha \in (0, 2)$. Show that $\{W(\tau(s)), s \geq 0\}$ is an $\alpha-$stable Lévy process.

Exercise 10-5 (3 points)

For the subordinator $\{T(t), t \geq 0\}$ with marginal density

$$f_{T(t)}(s) = \frac{t}{2\sqrt{\pi}} s^{-\frac{3}{2}} e^{-\frac{t^2}{4s}} \mathbb{I}\{s > 0\}$$

prove directly that its Laplace transform is $\mathbf{E}e^{-uT(t)} = \exp(-tu^{1/2})$. Show that $\{T(t), t \geq 0\}$ is a $\frac{1}{2}$-stable subordinator.

Hint: Differentiate the Laplace transform of $T(t)$ and solve the differential equation.